Potential theory in Euclidean spaces

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Gakuto International Series

Mathematical
Sciences
and
Applications

Volume 6

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January 1996

Gakkōtosho, Tokyo, Japan

Preface

This book is devoted to a study of properties of Riesz potentials of measures and functions in Lebesgue's L^p classes on the Euclidean space $\mathbf{R}^n$. This book, for the sake of completeness, begins with a self-contained introduction of measure theory, which follows the book of Federer.

Potential theory of Laplace operator is sometimes called the classical potential theory. The main thrust of the classical potential theory is to solve the Dirichlet problem. For this purpose, we give fundamental properties of harmonic and superharmonic functions in connection with Newtonian potentials. We here omit to treat Dirichlet problem for general partial differential equations of elliptic type, but prepare for several materials which may be useful for the study of general Dirichlet problems. Those things will belong to L^p-potential theory.

The main purpose of this book is to study pointwise behavior of Riesz potentials. In fact, continuity and differentiability properties are considered, together with fine limits, radial limits and L^q-mean limits.

The investigation of fine limits requires an analysis of exceptional sets, which can be conveniently described by Riesz and Bessel capacities. The notion of capacities is an extension of that of outer measures. Capacities give excellent tools in potential theory, but they are not easier to be treated by the reason that they fail to satisfy countable additivity.

Beppo Levi functions are referred to those functions defined on an open set in $\mathbf{R}^n$ whose distributional derivatives are in Lebesgue's L^p classes. Usually they are called Sobolev functions, associated with the name of the Soviet mathematician S.L.Sobolev. Beppo Levi functions are represented as integral forms, in various way. One way is to establish an extension of Sobolev's integral representation. Through integral representations, they can be treated in the same manner as Riesz potentials.

The final chapter is devoted to the study of boundary limits of various functions defined on half spaces. Potential type functions and Beppo Levi functions are shown to have various boundary limits in the weak sense, as a consequence of fine limit theorem. On the other hand, polyharmonic functions, together with monotone functions in the sense of Lebesgue, are shown to have tangential limits. Since solutions for variational problems should be monotone in our sense, the class of monotone functions is considerably wide.

An outline of this book is as follows. Chapter 1 is a short course of measure theory as mentioned above. The notion of Suslin (analytic) sets are introduced here and will be useful for capacitability result given in Chapter 2.

Chapter 2 deals with Riesz potentials of (Radon) measures. The semigroup property of Riesz kernels are shown by using Fourier transform. Riesz potentials may not be continuous anywhere, but they are finely continuous. Fine limit result will be obtained in connection with Riesz capacities, which are shown to be Choquet's capacity by use of minimax lemma. As a consequence, Suslin sets are capacitable. To give a solution to Dirichlet problem, we apply Gauss variational method through energy integral.

Chapter 3 gives fundamental properties of harmonic and superharmonic functions, for example, mean value property, maximum and minimum principles, Harnack's inequality and Poisson integral formula. As an application, we study the classical Dirichlet problem.

Chapter 4 and the following chapters belong to L^p-potential theory. Chapter 4 deals with α-potentials of L^p functions, which belong to L^q when $1/q = 1/p - \alpha/n > 0$. This is known as Sobolev's inequality, which will be shown by applying the Marcinkiewicz interpolation theorem. Further, restriction and inverse properties will be studied here.

Chapter 5 concerns with continuity and differentiability properties of α-potentials of L^p functions, in connection with (α, p)-capacities. In order to study the existence of radial limits, we need to modify the notion of thin sets to linear type, and further prepare contractive property for (α, p)-capacities.

In Chapter 6, we first give Sobolev's integral representation for Beppo Levi functions. To show the converse, we apply singular integral theory so as to see that the integral representations give Beppo Levi functions. Through the representations, we attach quasicontinuous functions to Beppo Levi functions, which are called BLD (Beppo-Levi-Deny) functions. BLD functions are ACL, that is, absolutely continuous along almost every line parallel to the coordinate axes.

Chapter 7 begins with defining Bessel kernels in the integral form; then Fourier transform of Bessel kernels are computed. The fractional order spaces are introduced by use of Poisson integrals, and we give the relationships between them and Bessel potential spaces.

The final chapter, Chapter 8, concerns with boundary limits of several types of functions. First, Green potentials in half spaces are treated and shown to have several limits as consequences of fine limit results. We also study the existence of curvilinear limits for BLD functions on half spaces. Next we give fundamental properties of polyharmonic functions; among them, our main purpose is to prepare for mean-value inequality for polyharmonic functions, which will be used for the study of tangential limits of polyharmonic BLD functions. Finally we are concerned with monotone BLD functions on half spaces which satisfy maximum and minimum principles.

In preparing this book, I often referred many books concerning potential theory and, especially, the lecture notes by Professor M. Ohtsuka given at Hiroshima University. I acknowledge gratefully his works on the subject. My thanks also go to Professor N. Kenmochi who encouraged me to write this book.

Contents

Notation

$\mathbf{N}$	the family of positive integers
$\mathbf{Z}$	the family of integers
$\mathbf{R}$	the real number field
$\mathbf{R}_+$	the family of positive numbers
$\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$	the extended real number field
$\mathbf{R}^n$	n-dimensional Euclidean space
$\mathbf{H}$	the half space $\{(x_1, x_2, ..., x_n) : x_1 > 0\}$
$\lvert x \rvert$	absolute value (norm) of $x \in \mathbf{R}^n$
$x \cdot y$	the inner product
$x' = (x_2, ..., x_n)$	for $x = (x_1, x_2, ..., x_n)$
$\overline{x} = (-x_1, x_2, ..., x_n)$	for $x = (x_1, x_2, ..., x_n)$
$B(x, r)$	open ball centered at x with radius r
$S(x, r) = \partial B(x, r)$	spherical surface centered at x with radius r
$\mathbf{B}$	the unit ball $B(0, 1)$
$\mathbf{S}$	the unit sphere $S(0, 1)$
$\emptyset$	the empty set
$\mathcal{B}$	the family of Borel sets
$A - B = \{x \in A : x \notin B\}$	difference of sets
$\mathcal{L}^n$	n-dimensional Lebesgue measure
$\lvert E \rvert = \mathcal{L}^n(E)$	n-dimensional measure of E
$\sigma_n = \lvert \mathbf{B} \rvert = \pi^{n/2}/(n/2)\Gamma(n/2)$	n-dimensional measure of the unit ball
$\omega_n = \lvert \mathbf{S} \rvert = 2\pi^{n/2}/\Gamma(n/2)$	the surface measure of $\mathbf{S}$
χ_A	characteristic function of A
$\mu\vert_A$	restriction of a measure μ to A
$\mu \times \nu$	product measure of μ and ν
$\mathcal{M}(E)$	the family of measures on E
$\mathcal{M} = \mathcal{M}(\mathbf{R}^n)$	the family of measures on $\mathbf{R}^n$
$\mathcal{M}_\mu$	the family of μ measurable sets
$V_a^b g$	total variation of g over the interval $[a, b]$
H_h	Hausdorff measure with a measure function h
$\mathcal{H}^n$	n-dimensional Hausdorff measure

$D_j = \partial/\partial x_j$	partial derivative with respect to j-th coordinate
$\nabla = (D_1, ..., D_n)$	gradient
∇^m	gradient iterated m times
$\lvert\lambda\rvert = \lambda_1 + \cdots + \lambda_n$	length of a multi-index $\lambda = (\lambda_1, ..., \lambda_n)$
$\begin{pmatrix} \mu \\ \nu \end{pmatrix}$	$= \begin{pmatrix} \mu_1 \\ \nu_1 \end{pmatrix} \cdots \begin{pmatrix} \mu_n \\ \nu_n \end{pmatrix}$
x^λ	${x_1}^{\lambda_1} \cdots {x_n}^{\lambda_n}$
$D^\lambda = D_1^{\lambda_1} \cdots D_n^{\lambda_n}$	partial derivative of order λ
$C_0(G)$	the space of continuous functions with compact support in G
$C_0^\infty(G)$	the space of infinitely differentiable functions with compact support in G
$\mathcal{D}'(G)$	the space of distributions on G
$\mathcal{S}$	the space of infinitely differentiable functions rapidly decreasing at ∞
$\mathcal{S}'$	the space of tempered distributions on $\mathbf{R}^n$
$\operatorname{dist}(x, E)$	distance of x to E
$\operatorname{dist}(E, F)$	distance between E and F
$M_\mu f$	maximal function of f with respect to μ
$\mu_{(p)}(f)$	L^p-norm of f with respect to μ
$L^p(\mu)$	the space of functions f with $\mu_{(p)}(f) < \infty$
S_μ	support of μ
$\mathcal{E}_\alpha(\mu, \nu)$	mutual α-energy
U_α	Riesz kernel of order α (α-kernel)
$\kappa_\alpha = \gamma_\alpha^{-1} U_\alpha$	normalized Riesz kernel of order α with $\gamma_\alpha = \pi^{n/2-\alpha}\Gamma(\alpha/2)/\Gamma((n-\alpha)/2)$
$U_\alpha \mu$	Riesz potential of order α (α-potential) of μ
$U_\alpha f = U_\alpha * f$	Riesz potential of order α (α-potential) of a function f
$\hat{f} = \mathcal{F} f$	Fourier transform of f
$\mathcal{F}^* f$	inverse Fourier transform of f
$L^p(G)$	the space of functions f on G such that $\lvert f\rvert^p$ is integrable with respect to Lebesgue measure
$\lVert f\rVert_{L^p(G)}$	L^p-norm of f in G with respect to Lebesgue measure
$\lVert f\rVert_p = \lVert f\rVert_{L^p(\mathbf{R}^n)}$	L^p-norm of f in $\mathbf{R}^n$
$Mf = M_\mu f$	maximal function of f with respect to $\mu = \mathcal{L}^n$
Δ	Laplace operator
Δ^m	Laplace operator iterated m times
C_α	outer Riesz capacity of order α (α-capacity)
c_α	inner Riesz capacity of order α (α-capacity)
$\mathbf{R}_+^{n+1}$	the half space $\mathbf{R}^n \times \mathbf{R}_+$
$G(x, y) = G_x(y)$	Green's function (kernel)
$P(x, y)$	Poisson kernel in $\mathbf{R}_+^{n+1}$
$P_t f = P_t * f$	Poisson integral of f in $\mathbf{R}_+^{n+1}$

μ_F	balayage of μ to F
G_α	Green's function of order α
$G_\alpha\mu$	Green's potential of μ of order α
γ_F	equilibrium measure of F
δ_a	Dirac measure at a
a.e.	almost everywhere
q.e.	quasi everywhere
$S_q(u,r)$	L^q-mean over $B(0,r)$
$C_{k,p}$	outer (k,p)-capacity
$c_{k,p}$	inner (k,p)-capacity
$C_{\alpha,p}$	outer (α,p)-capacity
$c_{\alpha,p}$	inner (α,p)-capacity
g_α	Bessel kernel of order α
$g_\alpha f = g_\alpha * f$	Bessel potential of f
$B_{\alpha,p}$	outer Bessel capacity of index (α,p)
$b_{\alpha,p}$	inner Bessel capacity of index (α,p)
$k_\lambda(x) = x^\lambda/\lvert x\rvert^n$	
$\tilde{k}_\lambda = D^\lambda U_{2m}$	
$k_{\lambda,\ell}$	remainder term of Taylor expansion of k_λ
sgn	sgn $t = +1$ if $t > 0$, $= 0$ if $t = 0$ and $= -1$ if $t < 0$
$BL_m(L^p(G))$	Beppo Levi space
$BL_m(L^p(G))^\bullet$	quotient Beppo Levi space by polynomials
$W^{m,p}(G)$	Sobolev space
$\lvert u\rvert_{m,p}$	$= \left(\sum_{\lvert\lambda\rvert=m} \lVert D^\lambda u\rVert_{L^p(G)}{}^p\right)^{1/p}$ seminorm for $u \in BL_m(L^p(G))$
$\lVert u\rVert_{m,p}$	$= \left(\sum_{\lvert\lambda\rvert\leqq m} \lVert D^\lambda u\rVert_{L^p(G)}{}^p\right)^{1/p}$ norm for $u \in W^{m,p}(G)$
Λ_α	Hölder space
$\Lambda_\alpha^{p,q}$	Lipschitz space
$T_\psi(\xi,a)$	the set $\{x = (x_1,...,x_n) : \psi(\lvert x-\xi\rvert) < ax_1\}$ for $\xi \in \partial\mathbf{H}$ and $a > 0$
$T_\gamma(\xi,a)$	$= T_\psi(\xi,a)$ with $\psi(r) = r^\gamma$
$T_1(\xi,a)$	cone with vertex at ξ
$\Delta_p u = -\mathrm{div}(\lvert\nabla u\rvert^{p-2}\nabla u)$	nonlinear Laplace operator

Chapter 1

Measure theory

Potential theory is based on measure theory. This chapter is a short course of measure theory, and we follow the book of Federer. The concept of Suslin (analytic) sets are most complicated; Suslin sets are shown to be capacitable in the next chapter.

1.1 Measures

In this book, we use $\mathbf{R}^n$ to denote the n-dimensional Euclidean space. The absolute value of a point $x = (x_1, ..., x_n) \in \mathbf{R}^n$ is defined by

$$|x| = \sqrt{{x_1}^2 + \cdots + {x_n}^2}.$$

The Cauchy-Schwartz inequality is of the form

$$\left| \sum_{j=1}^{n} x_j y_j \right| \leqq |x| \cdot |y|;$$

the sum is the inner product of $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, which is sometimes written as $x \cdot y$. For $x \in \mathbf{R}^n$ and $r > 0$, denote by $B(x, r)$ the open ball centered at x with radius r, that is,

$$B(x, r) = \{y \in \mathbf{R}^n : |x - y| < r\}.$$

We say that μ is an (outer) measure on $\mathbf{R}^n$ if

(i) $\mu(\emptyset) = 0$.

(ii) $0 \leqq \mu(A) \leqq \infty$ for all $A \subseteqq \mathbf{R}^n$

(iii) μ is countably subadditive, that is,

$$\mu(A) \leqq \sum_{j=1}^{\infty} \mu(A_j) \qquad \text{whenever } A \subseteqq \bigcup_{j=1}^{\infty} A_j.$$

We see readily from (iii) that μ is nondecreasing, that is,

$$\mu(A) \leqq \mu(B) \qquad \text{whenever } A \subseteqq B.$$

A set $A \subseteqq \mathbf{R}^n$ is (μ) measurable if

$$\mu(T) = \mu(T \cap A) + \mu(T - A) \qquad \text{for any } T \subseteqq \mathbf{R}^n. \tag{1.1}$$

For example, the empty set $\emptyset$ and the whole space $\mathbf{R}^n$ are measurable; further, note that if $\mu(A) = 0$, then A is measurable. By the countable subadditivity (iii), if

$$\mu(T) \geqq \mu(T \cap A) + \mu(T - A) \qquad \text{for any } T \subseteqq \mathbf{R}^n, \tag{1.1$'$}$$

then A is measurable.

The following is easy.

THEOREM 1.1. *If A is measurable, then the complement of A is also measurable.*

LEMMA 1.1. *The finite union and the finite intersection of measurable sets are measurable.*

PROOF. Let A_1 and A_2 be measurable sets. Then we have for $T \subseteqq \mathbf{R}^n$

$$\begin{aligned} \mu(T) &\geqq \mu(T \cap A_1) + \mu(T - A_1) \\ &\geqq \mu(T \cap A_1) + [\mu((T - A_1) \cap A_2) + \mu((T - A_1) - A_2)] \\ &\geqq \mu(T \cap (A_1 \cup A_2)) + \mu(T - (A_1 \cup A_2)), \end{aligned}$$

which implies that $A_1 \cup A_2$ is measurable. By induction, it follows that the finite unions of measurable sets are all measurable. Since $\mathbf{R}^n - A_1$ and $\mathbf{R}^n - A_2$ are measurable by Theorem 1.1, $(\mathbf{R}^n - A_1) \cup (\mathbf{R}^n - A_2)$ is measurable by the above considerations, so that Theorem 1.1 again implies that $A_1 \cap A_2$ is measurable.

LEMMA 1.2. *If A_j, $j = 1, 2, ..., N$, are measurable and mutually disjoint, then*

$$\mu\Big(\bigcup_{j=1}^{N} A_j\Big) = \sum_{j=1}^{N} \mu(A_j). \tag{1.2}$$

PROOF. Letting $T = A_1 \cup A_2$ and $A = A_1$ in (1.1), we have

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$$

since $(A_1 \cup A_2) \cap A_1 = A_1$ and $(A_1 \cup A_2) - A_1 = A_2$. Now we have (1.2) by induction.

THEOREM 1.2. *Let $\{A_j\}$ be a sequence of measurable sets in $\mathbf{R}^n$. If it is mutually disjoint, then*

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j). \tag{1.3}$$

PROOF. If we set $B_N = \bigcup_{j=1}^{N} A_j$, then we see from Lemma 1.1 that B_N is measurable. Since $\{A_j\}$ is mutually disjoint, Lemma 1.2 gives

$$\mu(\bigcup_{j=1}^{\infty} A_j) \geqq \mu(B_N) = \sum_{j=1}^{N} \mu(A_j),$$

so that

$$\mu(\bigcup_{j=1}^{\infty} A_j) \geqq \sum_{j=1}^{\infty} \mu(A_j).$$

Now the countable subadditivity (iii) yields (1.3).

THEOREM 1.3. *If $\{A_j\}$ is a nondecreasing sequence of measurable sets in $\mathbf{R}^n$, then*

$$\mu(\bigcup_{j=1}^{\infty} A_j) = \lim_{j\to\infty} \mu(A_j). \tag{1.4}$$

PROOF. If we set $B_j = A_j - A_{j-1}$ with $A_0 = \emptyset$, then we see from Lemma 1.1 that $B_j = A_j \cap (\mathbf{R}^n - A_{j-1})$ is measurable. Since $\{B_j\}$ is mutually disjoint, Theorem 1.2 gives

$$\mu(\bigcup_{j=1}^{\infty} A_j) = \mu(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \mu(B_j) = \lim_{j\to\infty} \mu(A_j),$$

as required.

THEOREM 1.4. *Let $\{A_j\}$ be a nonincreasing sequence of measurable sets in $\mathbf{R}^n$. If $\mu(A_1) < \infty$, then*

$$\mu(\bigcap_{j=1}^{\infty} A_j) = \lim_{j\to\infty} \mu(A_j). \tag{1.5}$$

PROOF. If we set $C_j = A_j - A_{j+1}$, then we see from Lemma 1.1 that C_j is measurable. Since $\{C_j\}$ is mutually disjoint, Theorems 1.2 and 1.3 give

$$\begin{aligned} \mu(A_1) &= \mu((\bigcap_{j=1}^{\infty} A_j) \cup (\bigcup_{j=1}^{\infty} C_j)) = \mu(\bigcap_{j=1}^{\infty} A_j) + \sum_{j=1}^{\infty} \mu(C_j) \\ &= \mu(\bigcap_{j=1}^{\infty} A_j) + \lim_{N\to\infty} [\mu(A_1) - \mu(A_N)]. \end{aligned}$$

Hence (1.5) is obtained since $\mu(A_1) < \infty$.

For a set $X \subseteqq \mathbf{R}^n$, denote the restriction of μ to X by $\mu|_X$, that is,

$$\mu|_X(A) = \mu(X \cap A) \qquad \text{for } A \subseteqq \mathbf{R}^n.$$

Then $\mu|_X$ is trivially a measure on $\mathbf{R}^n$. We say that $\mu|_X$ is a measure on X.

LEMMA 1.3. *If A is μ measurable, then A is also $\mu|_T$ measurable for every $T \subseteqq \mathbf{R}^n$.*

THEOREM 1.5. *If $\{A_j\}$ is a countable family of measurable sets, then $\bigcup_{j=1}^{\infty} A_j$ and $\bigcap_{j=1}^{\infty} A_j$ are also measurable.*

PROOF. Suppose $\mu(T) < \infty$. Then we see from Theorems 1.3 and 1.4 that

$$\begin{aligned} \mu(T \cap \bigcup_{j=1}^{\infty} A_j) + \mu(T - \bigcup_{j=1}^{\infty} A_j) &= \lim_{m\to\infty} \mu|_T(\bigcup_{j=1}^{m} A_j) + \lim_{m\to\infty} \mu|_T(\mathbf{R}^n - \bigcup_{j=1}^{m} A_j) \\ &= \mu|_T(\mathbf{R}^n) = \mu(T), \end{aligned}$$

which shows that $\bigcup_{j=1}^{\infty} A_j$ is measurable. The intersection of countably many measurable sets is also seen to be measurable.

1.2 Borel sets

A family $\mathcal{A}$ of sets is called a σ-algebra if

(i) if $A \in \mathcal{A}$, then $\mathbf{R}^n - A \in \mathcal{A}$;

(ii) if $A_j \in \mathcal{A}$, then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$.

By (i) and (ii), we see that

(iii) if $A_j \in \mathcal{A}$, then $\bigcap_{j=1}^{\infty} A_j \in \mathcal{A}$.

The smallest σ-algebra containing all compact sets is called the family of Borel sets, which is denoted by $\mathcal{B}$. Each element of $\mathcal{B}$ is called a Borel set in $\mathbf{R}^n$.

Denote by $\mathcal{M}_\mu$ the family of all μ measurable sets. Then, in view of Theorems 1.1 and 1.5, we see that $\mathcal{M}_\mu$ is a σ-algebra.

LEMMA 2.1. *If all compact sets are measurable, then all Borel sets are measurable; that is, $\mathcal{B} \subseteqq \mathcal{M}_\mu$.*

LEMMA 2.2. *If $\mathcal{A}$ satisfies (ii) and (iii), then the subfamily*

$$\tilde{\mathcal{A}} = \{A : A \in \mathcal{A} \quad \text{and} \quad \mathbf{R}^n - A \in \mathcal{A}\}$$

satisfies (ii).

To show this, let $A_j \in \tilde{\mathcal{A}}$. Then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ by (ii) and

$$\mathbf{R}^n - \bigcup_{j=1}^{\infty} A_j = \bigcap_{j=1}^{\infty} (\mathbf{R}^n - A_j) \in \mathcal{A}$$

by (iii), and hence $\tilde{\mathcal{A}}$ satisfies (ii).

THEOREM 2.1. *Let μ be a measure on $\mathbf{R}^n$ for which all compact sets are measurable. If B is a Borel set with $\mu(B) < \infty$, then for any $\varepsilon > 0$ there exists a compact set $K \subseteqq B$ such that $\mu(B - K) < \varepsilon$, that is,*

$$\mu(B) = \sup\{\mu(K) : K \subseteqq B,\ K : \textit{compact}\}.$$

PROOF. Consider the family $\mathcal{A}$ of sets A such that for any $\varepsilon > 0$ there exists a compact set $K \subseteqq A$ with $\mu|_B(A - K) < \varepsilon$. First we show that $\mathcal{A}$ satisfies (ii) and (iii). Let $A_j \in \mathcal{A}$ for each positive integer j. Given $\varepsilon > 0$, take compact sets K_j for which

$$\mu|_B(A_j - K_j) < 2^{-j}\varepsilon.$$

Note that

$$\mu|_B(\bigcup_{j=1}^{\infty} A_j - \bigcup_{j=1}^{\infty} K_j) \leqq \mu|_B(\bigcup_{j=1}^{\infty}(A_j - K_j)) < \sum_{j=1}^{\infty} 2^{-j}\varepsilon = \varepsilon,$$

$$\mu|_B(\bigcap_{j=1}^{\infty} A_j - \bigcap_{j=1}^{\infty} K_j) \leqq \mu|_B(\bigcup_{j=1}^{\infty}(A_j - K_j)) < \varepsilon$$

and, with the aid of Theorem 1.5,

$$\lim_{N\to\infty} \mu|_B(\bigcup_{j=1}^{\infty} A_j - \bigcup_{j=1}^{N} K_j) = \mu|_B(\bigcup_{j=1}^{\infty} A_j - \bigcup_{j=1}^{\infty} K_j) < \varepsilon.$$

Hence $\mathcal{A}$ satisfies (ii) and (iii). Now, in view of Lemma 2.2, $\tilde{\mathcal{A}}$ satisfies (i) and (ii). Clearly, $\mathcal{A}$ contains all compact sets, and then it contains all open sets with the aid of Theorem 1.3, since open sets are K_σ-sets. Hence $\tilde{\mathcal{A}}$ contains all compact sets. Thus all Borel sets are contained in $\tilde{\mathcal{A}}$; in particular, $B \in \tilde{\mathcal{A}}$, as required.

THEOREM 2.2. *Let μ be a measure on $\mathbf{R}^n$ for which all compact sets are measurable. If B is a Borel set with a countable covering $\{G_j\}$ of open sets such that $\mu(G_j) < \infty$, then for any $\varepsilon > 0$ there exists an open set $G \supseteqq B$ for which*

$$\mu(G - B) < \varepsilon.$$

PROOF. Let $\varepsilon > 0$ be given. For each positive integer j, take a compact set $K_j \subseteqq G_j - B$ such that

$$\mu((G_j - B) - K_j) < 2^{-j}\varepsilon.$$

Noting that $B \cap G_j \subseteqq G_j - K_j$, we have only to consider

$$G = \bigcup_{j=1}^{\infty} (G_j - K_j).$$

A measure μ on $\mathbf{R}^n$ is called regular if for any $A \subseteqq \mathbf{R}^n$ there exists a measurable set B such that $A \subseteqq B$ and

$$\mu(A) = \mu(B). \tag{2.1}$$

LEMMA 2.3. *Let μ be a measure on $\mathbf{R}^n$. For any set $A \subseteqq \mathbf{R}^n$, define*

$$\gamma(A) = \inf\{\mu(B) : A \subseteqq B,\ B : \textit{measurable}\}.$$

Then γ is regular and $\mathcal{M}_\mu \subseteqq \mathcal{M}_\gamma$.

PROOF. Let $T \subseteqq \mathbf{R}^n$ and take a measurable set A such that

$$\gamma(T) = \mu(A), \qquad T \subseteqq A. \tag{2.2}$$

If B is measurable, then we have

$$\gamma(T \cap B) + \gamma(T - B) \leqq \mu(A \cap B) + \mu(A - B) = \mu(A) = \gamma(T),$$

which implies that B is γ measurable. Moreover, (2.2) implies that γ is regular.

THEOREM 2.3. *Let μ be regular. If $\{A_j\}$ is a nondecreasing sequence of sets in $\mathbf{R}^n$, then*

$$\mu(\bigcup_{j=1}^{\infty} A_j) = \lim_{j\to\infty} \mu(A_j).$$

PROOF. Take a measurable set B_j containing A_j such that

$$\mu(B_j) = \mu(A_j),$$

and set

$$C_i = \bigcap_{j=i}^{\infty} B_j.$$

Then $A_i \subseteqq C_i \subseteqq B_i$ and $C_i \subseteqq C_{i+1}$. Hence Theorem 1.3 implies that

$$\mu(\bigcup_{i=1}^{\infty} A_i) \leqq \mu(\bigcup_{i=1}^{\infty} C_i) = \lim_{i\to\infty} \mu(C_i) = \lim_{i\to\infty} \mu(A_i).$$

Thus the required equality now follows.

A measure μ on $\mathbf{R}^n$ is said to be a Borel measure if all Borel sets are measurable. A Borel regular measure μ on $\mathbf{R}^n$ is called a Radon measure if $\mu(K) < \infty$ for any compact set K in $\mathbf{R}^n$.

In view of Theorem 2.2, we have the following.

THEOREM 2.4. *If μ is a Radon measure on $\mathbf{R}^n$, then for any set $A \subseteqq \mathbf{R}^n$,*

$$\mu(A) = \inf\{\mu(G) : G \text{ is open and } A \subseteqq G\}. \tag{2.3}$$

1.3 Suslin sets

Let $\mathbf{N}$ be the family of all positive integers, and denote by $\mathcal{N}$ the family of all sequence of positive integers. We can define a metric on $\mathcal{N}$ by setting

$$\rho(\{a_j\}, \{b_j\}) = \sum_{j=1}^{\infty} 2^{-j} \frac{|a_j - b_j|}{1 + |a_j - b_j|} .$$

More precisely,

(i) $\rho(\{a_j\}, \{b_j\}) \geqq 0$ and $\rho(\{a_j\}, \{b_j\}) = 0$ if and only if $\{a_j\} = \{b_j\}$;

(ii) $\rho(\{a_j\}, \{b_j\}) = \rho(\{b_j\}, \{a_j\})$;

(iii) $\rho(\{a_j\}, \{c_j\}) \leqq \rho(\{a_j\}, \{b_j\}) + \rho(\{b_j\}, \{c_j\})$.

Note here that (i), (ii) are trivial and (iii) follows if one notes that the function:

$$t \to \frac{t}{1+t}$$

is increasing on the interval $(0, \infty)$.

LEMMA 3.1. *The metric space $(\mathcal{N}, \rho)$ is complete and separable.*

PROOF. Let $\{a_j^{(i)}\} \in \mathcal{N}$ be a Cauchy sequence in $\mathcal{N}$. Then $\{a_j^{(i)}\}_{i=1}^{\infty}$ is a Cauchy sequence for each j, and thus it converges to $a_j^{(\infty)}$. Note that $\{a_j^{(i)}\}$ converges to $\{a_j^{(\infty)}\}$. On the other hand, for $\{a_j\} \in \mathcal{N}$, consider the sequences $\{b_j^{(m)}\}$ such that

$$b_j^{(m)} = \begin{cases} a_j & \text{when } j \leqq m, \\ 1 & \text{otherwise.} \end{cases}$$

It is not difficult to see that $\{b_j^{(m)}\} \to \{a_j\}$ as $m \to \infty$ and further that the family of all finite sequences of positive integers are countable.

Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that $f : X \to Y$ is locally Lipschitzian if each point x of X has an open neighborhood G such that

$$d_Y(f(y), f(z)) \leqq M d_X(y, z) \qquad \text{whenever } y,\ z \in G.$$

Proposition 3.1. *If (X, d_X) is a complete, separable and nonempty metric space, then there exists a locally Lipschitzian map g of $\mathcal{N}$ onto X.*

Proof. Since X is separable, we can take $\{x_{j_1}\} \subseteqq X$ such that

$$X = \bigcup_{j_1 \in \mathbf{N}} E(j_1), \qquad E(j_1) = \{x \in X : d_X(x, x_{j_1}) \leqq 2^{-1}\}.$$

Similarly, for each j_1 we can take $\{x_{j_1,j_2}\} \subseteqq E(j_1)$ such that

$$E(j_1) = \bigcup_{j_2 \in \mathbf{N}} E(j_1, j_2), \qquad E(j_1, j_2) = \{x \in E(j_1) : d_X(x, x_{j_1,j_2}) \leqq 2^{-2}\}.$$

Repeating this process, we obtain $\{x_{j_1,j_2,...,j_m}\}$ and $\{E(j_1, j_2, ..., j_m)\}$ such that

$$E(j_1, j_2, ..., j_m) = \bigcup_{j_{m+1} \in \mathbf{N}} E(j_1, j_2, ..., j_m, j_{m+1})$$

and

$$E(j_1, j_2, ..., j_{m+1}) = \{x \in E(j_1, j_2, ..., j_m) : d_X(x, x_{j_1,j_2,...,j_m,j_{m+1}}) \leqq 2^{-m-1}\}.$$

Then one notes that for each $\{j_m\} \in \mathcal{N}$,

$$E(j_1) \supseteqq E(j_1, j_2) \supseteqq E(j_1, j_2, j_3) \supseteqq \cdots$$

and the intersection

$$E(j_1) \cap E(j_1, j_2) \cap E(j_1, j_2, j_3) \cap \cdots$$

has at most one point $g(j_1, j_2, ...)$ in X, which is the limit point of $\{x_{j_1}, x_{j_1,j_2}, ...\}$. Thus we obtain the map g of $\mathcal{N}$ to X. It is easy to see that g is onto. If $2^{-k-2} \leqq \rho(\{i_m\}, \{j_m\}) < 2^{-k-1}$, then

$$i_m = j_m \qquad \text{for } m \leqq k,$$

so that

$$d_X(g(\{i_m\}), g(\{j_m\})) < 2^{-k+1} \leqq 8\rho(\{i_m\}, \{j_m\}),$$

which implies that g is locally Lipschitzian.

PROPOSITION 3.2. *Let (X, d_X) be as above. If in addition X has no isolated point, then there exists a subset Γ of X which is homeomorphic to $\mathcal{N}$.*

PROOF. First we take a mutually disjoint family $\{U(j_1)\}$ of nonempty open sets with diameter less than 2^{-1}. Next we take a mutually disjoint family $\{U(j_1, j_2)\}$ of nonempty open sets with diameter less than 2^{-2} for which

$$U(j_1) \supseteqq \bigcup_{j_2 \in \mathbf{N}} \overline{U(j_1, j_2)}.$$

Repeating this process, we obtain a mutually disjoint family $\{U(j_1, j_2, ..., j_m)\}$ of nonempty open sets with diameter less than 2^{-m} for which

$$U(j_1, j_2, ..., j_m) \supseteqq \bigcup_{j_{m+1} \in \mathbf{N}} \overline{U(j_1, j_2, ..., j_m, j_{m+1})}.$$

Now define

$$\Gamma = \bigcap_{m \in \mathbf{N}} \bigcup_{s \in \mathbf{N}^m} U(s).$$

Since the intersection

$$U(j_1) \cap U(j_1, j_2) \cap U(j_1, j_2, j_3) \cap \cdots$$

has at most one point $g(j_1, j_2, ...)$ in Γ as in the proof of Proposition 3.1. Noting that g is locally Lipschitzian and one-to-one, we conclude the proof.

When $X = \mathbf{R}$, in the above proof, we let $U(j)$ be open intervals such that

$$\mathbf{R} = \bigcup_{j_1 \in \mathbf{N}} \overline{U(j_1)}$$

and

$$U(j_1, j_2, ..., j_m) = \bigcup_{j_{m+1} \in \mathbf{N}} \overline{U(j_1, j_2, ..., j_m, j_{m+1})}.$$

Then we have the following.

PROPOSITION 3.3. *There exists a subset Γ of $\mathbf{R}$ such that Γ is homeomorphic to $\mathcal{N}$ and $\mathbf{R} - \Gamma$ is countable.*

Consider the projection

$$p : \mathbf{R}^n \times \mathcal{N} \to \mathbf{R}^n,$$

that is,

$$p(x, s) = x \qquad \text{for } x \in \mathbf{R}^n \text{ and } s \in \mathcal{N}.$$

By a Suslin subset of $\mathbf{R}^n$ we mean the p image of a closed subset of $\mathbf{R}^n \times \mathcal{N}$.

PROPOSITION 3.4. *If f is a continuous mapping of $\mathcal{N}$ in $\mathbf{R}^n$, then the image of f, $f(\mathcal{N})$, is a Suslin set in $\mathbf{R}^n$.*

In fact, $F = \{(x,s) \in \mathbf{R}^n \times \mathcal{N} : x = f(s)\}$ is closed and

$$f(\mathcal{N}) = p(F).$$

Next we show that every Borel set is a Suslin set. To do this, consider the family

$$\mathcal{A} = \{p(F) : F \subseteqq \mathbf{R}^n \times \mathcal{N} \quad \text{is closed and } p|_F \text{ is one-to-one}\}.$$

LEMMA 3.1. *If G is an open set in $\mathbf{R}^n$, then $G \in \mathcal{A}$.*

PROOF. Consider the closed set in $\mathbf{R}^n \times \mathbf{R}$:

$$A = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} : t \operatorname{dist}(x, \mathbf{R}^n - G) = 1\}.$$

Note here that $\mathcal{N} \times \mathcal{N}$ is homeomorphic to $\mathcal{N}$ and, by Proposition 3.2, $\mathcal{N}$ is homeomorphic to Γ for which $\mathbf{R} - \Gamma$ is countable. Hence, considering one-to-one maps:

$$\mathbf{R} = \Gamma \cup (\mathbf{R} - \Gamma) \to \mathcal{N} \times \mathcal{N} \to \mathcal{N},$$

we can find a closed subset B of $\mathcal{N}$ and a one-to-one continuous map ψ of B onto $\mathbf{R}$. Define

$$C = \{(x,s) \in \mathbf{R}^n \times \mathcal{N} : s \in B, (x, \psi(s)) \in A\}.$$

Then C is closed in $\mathbf{R}^n \times \mathcal{N}$, $p|_C$ is one-to-one and $p(C) = G$.

LEMMA 3.2. *If F_j is an element of $\mathcal{A}$ which is mutually disjoint, then $\bigcup_j F_j \in \mathcal{A}$.*

PROOF. For each positive integer j, let

$$\mathcal{N}_j = \{s \in \mathcal{N} : s_1 = j\},$$

which is homeomorphic to $\mathcal{N}$. By assumption, take a closed set $C_j \subseteqq \mathbf{R}^n \times \mathcal{N}_j$ so that $p|_{C_j}$ is one-to-one and $p(C_j) = F_j$. Consider

$$D = \bigcup_{j=1}^{\infty} C_j.$$

Note then that D is closed in $\mathbf{R}^n \times \mathcal{N}$, $p|_D$ is one-to-one and

$$p(D) = \bigcup_{j=1}^{\infty} F_j.$$

LEMMA 3.3. *If F_j belongs to $\mathcal{A}$, then $F = \bigcap_j F_j \in \mathcal{A}$.*

PROOF. For each positive integer j, take a closed set $C_j \subsetneqq \mathbf{R}^n \times \mathcal{N}$ so that $p|_{C_j}$ is one-to-one and $p(C_j) = F_j$. Considering the homeomorphism $\mathcal{N}^{\mathbf{N}}$ onto $\mathcal{N}$, we see that F is the one to one image of the closed set

$$\{(x, s_1, s_2, ...) : (x, s_j) \in C_j \quad \text{for each } j \in \mathbf{N}\}.$$

In view of Lemmas 3.1, 3.2 and 3.3, we have the following conclusion.

THEOREM 3.1. *Every Borel set in* $\mathbf{R}^n$ *is a Suslin set in* $\mathbf{R}^n$.

THEOREM 3.2. *Let* μ *be a measure on* $\mathbf{R}^n$ *for which all compact sets are measurable, and* S *be a Suslin set. If* $\mu(T) < \infty$ *and* $\varepsilon > 0$, *then there exists a compact set* $F \subsetneqq S$ *for which*

$$\mu(T \cap S - F) < \varepsilon.$$

PROOF. Take a closed set $Z_0 \subsetneqq \mathbf{R}^n \times \mathcal{N}$ such that

$$p(Z_0) = S.$$

By considering $\mu|T$, we may assume that $\mu(\mathbf{R}^n) < \infty$. Define a regular measure:

$$\gamma(A) = \inf\, \{\mu(B) : A \subsetneqq B, B : \text{open}\},$$

with the aid of Lemma 2.3. According to Theorem 2.3, we can find $\{m_i\} \in \mathcal{N}$ such that

$$\gamma(p(Z_{i-1})) < \gamma(p(Z_i)) + 2^{-i}\varepsilon, \qquad Z_i = \{(x, s) \in Z_{i-1} : s_i \leqq m_i\},$$

since

$$Z_{i-1} = \bigcup_{j=1}^{\infty} \{(x, s) \in Z_{i-1} : s_i \leqq j\}.$$

Then note that

$$\gamma(S) < \gamma(p(Z_i)) + \varepsilon$$

for every i. Setting $C = \bigcap_{i=1}^{\infty} \overline{p(Z_i)}$, we see from Theorem 1.4 that

$$\gamma(C) = \lim_{i \to \infty} \gamma(\overline{p(Z_i)}) > \gamma(S) - \varepsilon.$$

Further observe that

$$K = \{s \in \mathcal{N} : s_i \leqq m_i\}$$

is compact and that

$$\bigcap_{i=1}^{\infty} Z_i = Z_0 \cap (\mathbf{R}^n \times K).$$

If we show that

$$C = p(Z_0 \cap (\mathbf{R}^n \times K)), \tag{3.1}$$

then it follows that $C \subseteqq S$, and hence the proof is completed. Since

$$C \supseteqq p(Z_0 \cap (\mathbf{R}^n \times K))$$

is trivial, we have only to show the opposite inclusion. For this purpose, let

$$x \in \mathbf{R}^n - p(Z_0 \cap (\mathbf{R}^n \times K)).$$

Then $(\{x\} \times K) \cap Z_0 = \emptyset$. Since Z_0 is closed and $\{x\} \times K$ is compact, we can find open sets V and W such that $x \in V$, $K \subseteqq W$ and $(V \times W) \cap Z_0 = \emptyset$. If $\mathrm{dist}(K, \mathcal{N} - W) > 2^{-i}$, then it follows that

$$Z_i \subseteqq (\mathbf{R}^n \times W) - (V \times W) = (\mathbf{R}^n - V) \times W,$$

so that

$$p(Z_i) \subseteqq \mathbf{R}^n - V.$$

This implies that $x \notin \overline{p(Z_i)}$, which proves (3.1).

COROLLARY 3.1. *Let μ be a measure on $\mathbf{R}^n$ for which all compact sets are measurable. Then any Suslin set is measurable.*

1.4 Measurable functions

Let μ be a measure on $\mathbf{R}^n$. An extended real-valued function f is called (μ) measurable if

$$\{x \in \mathbf{R}^n : f(x) > t\}$$

is (μ) measurable for every real number t.

THEOREM 4.1. *The following statements are equivalent.*

(1) *f is measurable on $\mathbf{R}^n$.*

(2) *$\{x \in \mathbf{R}^n : f(x) \geqq t\}$ is measurable for every $t \in \mathbf{R}$.*

(3) *$\{x \in \mathbf{R}^n : f(x) < t\}$ is measurable for every $t \in \mathbf{R}$.*

(4) *$\{x \in \mathbf{R}^n : f(x) \leqq t\}$ is measurable for every $t \in \mathbf{R}$.*

For example, to show the implication (1) $\Rightarrow$ (2), it suffices to see that

$$\{x \in \mathbf{R}^n : f(x) \geqq t\} = \bigcap_{j=1}^{\infty} \{x \in \mathbf{R}^n : f(x) > t - 1/j\}.$$

The remaining cases are left to the reader.

Corollary 4.1. *A function f is measurable on $\mathbf{R}^n$ if and only if $f^{-1}(G)$ is measurable for every open set G in the extended real line $\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$.*

Corollary 4.2. *Let φ be continuous on $\overline{\mathbf{R}}$ in the extended sense. If a function f is measurable on $\mathbf{R}^n$, then $\varphi \circ f$ is also measurable on $\mathbf{R}^n$.*

Theorem 4.2. *A function f is measurable on $\mathbf{R}^n$ if and only if*

$$\mu(T) \geqq \mu(\{x \in T : f(x) \leqq s\}) + \mu(\{x \in T : f(x) \geqq t\}) \tag{4.1}$$

whenever $T \subseteqq \mathbf{R}^n$ and $-\infty < s < t < \infty$.

Proof. Set $E(s) = \{x \in \mathbf{R}^n : f(x) \leqq s\}$. If f is measurable on $\mathbf{R}^n$, then we have for any $T \subseteqq \mathbf{R}^n$ and $s < t$,

$$\begin{aligned} \mu(T) &\geqq \mu(T \cap E(s)) + \mu(T - E(s)) \\ &= \mu(\{x \in T : f(x) \leqq s\}) + \mu(\{x \in T : f(x) > s\}) \\ &\geqq \mu(\{x \in T : f(x) \leqq s\}) + \mu(\{x \in T : f(x) \geqq t\}). \end{aligned}$$

Conversely, suppose $\mu(T) < \infty$ and (4.1) holds. Set

$$A_j = \{x \in T : s + (j+1)^{-1} \leqq f(x) \leqq s + j^{-1}\}.$$

By (4.1), we have

$$\mu(\bigcup_{j \in \Delta} A_j) = \sum_{j \in \Delta} \mu(A_j)$$

if Δ is a finite set of positive integers whose elements are all even or all odd. Hence it follows that

$$\sum_j \mu(A_j) \leqq 2\mu(T) < \infty.$$

Consequently, for $\varepsilon > 0$, we can find j_0 so that

$$\mu(\{x \in T : s < f(x) \leqq s + j_0^{-1}\}) \leqq \sum_{j \geqq j_0} \mu(A_j) < \varepsilon.$$

We now find

$$\begin{aligned} \mu(T \cap E(s)) + \mu(T - E(s)) - \varepsilon &\leqq \mu(\{x \in T : f(x) \leqq s\}) \\ &\quad + \mu(\{x \in T : f(x) > s + j_0^{-1}\}) \leqq \mu(T), \end{aligned}$$

which implies that $E(s)$ is measurable.

COROLLARY 4.3. *Any open set in* $\mathbf{R}^n$ *is measurable if and only if*

$$\mu(A \cup B) = \mu(A) + \mu(B) \tag{4.2}$$

whenever $A \subseteqq \mathbf{R}^n$, $B \subseteqq \mathbf{R}^n$ *and* $\operatorname{dist}(A, B) > 0$.

PROOF. Suppose $\mu(T) < \infty$, G is open and (4.2) holds. Consider the sets

$$A_j = \{x \in T : (j+1)^{-1} \leqq \operatorname{dist}(x, G) \leqq j^{-1}\}.$$

As in the above proof, we see that

$$\sum_j \mu(A_j) \leqq 2\mu(T) < \infty$$

and, for $\varepsilon > 0$, find j_0 so that

$$\mu(T \cap G) + \mu(T - G) - \varepsilon \leqq \mu(T \cap G)) + \mu(T - \bigcup_{j \geqq j_0} A_j) \leqq \mu(T),$$

which implies that G is measurable.

Conversely suppose any open set is measurable. Let $A \subseteqq \mathbf{R}^n$, $B \subseteqq \mathbf{R}^n$ and $\operatorname{dist}(A, B) > 0$. Then, taking an open set G such that $A \subseteqq G$ and $B \subseteqq \mathbf{R}^n - G$, we have

$$\mu(A \cup B) \geqq \mu((A \cup B) \cap G) + \mu((A \cup B) - G) = \mu(A) + \mu(B)$$

and the proof is now completed.

THEOREM 4.3. *Let* f *be a measurable function on* $\mathbf{R}^n$. *If* S *is a Suslin set in* $\mathbf{R}$, *then* $f^{-1}(S)$ *is measurable.*

PROOF. First note that

(4.3) $f^{-1}(B)$ is measurable whenever B is a Borel set in $\mathbf{R}$.

To see this, we have only to show that the family

$$\mathcal{A} = \{A \subseteqq \mathbf{R} : f^{-1}(A) \text{ is measurable}\}$$

is a σ-algebra containing all intervals. For any set A in $\mathbf{R}^n$, define a measure ν_A on $\mathbf{R}$ by setting

$$\nu_A(T) = \mu(A \cap f^{-1}(T)).$$

Then, for $B \subseteqq \mathbf{R}$, we infer that

(4.4) $f^{-1}(B)$ is measurable if and only if B is ν_A measurable for any $A \subseteqq \mathbf{R}^n$.

Now, any Borel set $B \subseteqq \mathbf{R}$ is ν_A measurable, because $f^{-1}(B)$ is measurable by (4.3). Thus Corollary 3.1 implies that S is ν_A measurable for every Suslin set $S \subseteqq \mathbf{R}$, so that it follows from (4.4) that $f^{-1}(S)$ is measurable.

THEOREM 4.4. *Let $\{f_j\}$ be a sequence of measurable functions on $\mathbf{R}^n$. Then $\limsup_{j\to\infty} f_j(x)$ and $\liminf_{j\to\infty} f_j(x)$ are measurable on $\mathbf{R}^n$.*

To show this, note that

$$\left\{x \in \mathbf{R}^n : \limsup_{j\to\infty} f_j(x) > t\right\} = \bigcup_{m=1}^{\infty}\left(\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty}\{x \in \mathbf{R}^n : f_j(x) > t + 1/m\}\right).$$

For a set $E \subseteqq \mathbf{R}^n$, denote by χ_E the characteristic function of E.

THEOREM 4.5. *Let f be a nonnegative measurable function on $\mathbf{R}^n$, and $\{r_j\}$ be a sequence of positive numbers such that*

$$\lim_{j\to\infty} r_j = 0 \qquad \text{and} \qquad \sum_{j=1}^{\infty} r_j = \infty.$$

Then there exists a sequence $\{E_j\}$ of measurable sets such that

$$f(x) = \sum_{j=1}^{\infty} r_j \chi_{E_j}(x) \qquad \text{for } x \in \mathbf{R}^n.$$

PROOF. First set

$$E_1 = \{x \in \mathbf{R}^n : f(x) > r_1\}.$$

Now we define $\{E_j\}$ inductively by

$$E_i = \left\{x \in \mathbf{R}^n : f(x) > r_i + \sum_{j=1}^{i-1} r_j \chi_{E_j}(x)\right\}.$$

For each $x \in \mathbf{R}^n$, let $\{j : x \in E_j\} = \{j_1, j_2, ...\}$, $j_1 < j_2 < \cdots$. If $j_i < m < j_{i+1}$, then we see that

$$r_{j_1} + r_{j_2} + \cdots + r_{j_i} < f(x) \leqq r_{j_1} + r_{j_2} + \cdots + r_{j_i} + r_m,$$

which implies that

$$f(x) = r_{j_1} + r_{j_2} + \cdots = \sum_{j=1}^{\infty} r_j \chi_{E_j}(x).$$

THEOREM 4.6 (Lusin's theorem). *Let μ be Borel regular and f be measurable on $\mathbf{R}^n$. If A is a measurable set in $\mathbf{R}^n$ with $\mu(A) < \infty$ and $\mu(\{x \in A : |f(x)| = \infty\}) = 0$,* *then for any $\varepsilon > 0$ there exists a compact set K such that $K \subseteqq A$, $\mu(A - K) < \varepsilon$ and $f|_K$ is continuous.*

Proof. For each positive integer $i \in \mathbf{N}$ and each integer $j \in \mathbf{Z}$, $\mathbf{Z} = \{0, \pm 1, \pm 2, ...\}$, set

$$B_{i,j} = (j/i, (j+1)/i].$$

Considering

$$A_{i,j} = A \cap f^{-1}(B_{i,j}),$$

for any $\varepsilon > 0$ we take compact sets $K_{i,j}$, with the aid of Theorem 2.1, such that $K_{i,j} \subseteqq A_{i,j}$ and

$$\mu(A_{i,j} - K_{i,j}) < \varepsilon 2^{-i-j}.$$

Then we have

$$\mu(A - \bigcup_{j \in \mathbf{Z}} K_{i,j}) < 3\varepsilon 2^{-i},$$

so that we can find $j(i)$ for which

$$\mu(A - F_i) < 3\varepsilon 2^{-i}, \qquad F_i = \bigcup_{|j| \leqq j(i)} K_{i,j}.$$

Choosing $r_{i,j} \in B_{i,j}$, we define a function

$$g_i(x) = r_{i,j} \qquad \text{for } x \in K_{i,j},\ |j| \leqq j(i).$$

Since $|g_i(x) - f(x)| < i^{-1}$ for $x \in F_i$, we see that g_i is uniformly convergent to f on $F = \bigcap_i F_i$ with

$$\mu(A - F) \leqq \sum_i \mu(A - F_i) < \varepsilon.$$

Clearly, each g_i is continuous as a function on F, and so is $f|_F$.

Theorem 4.7 (Egoroff's theorem). *Let $\{f_j\}$ be a sequence of measurable functions on $\mathbf{R}^n$ which converges to g on a set $A \subseteqq \mathbf{R}^n$. If $\mu(A) < \infty$ and $\varepsilon > 0$, then there exists a measurable set $B \subseteqq A$ such that $\mu(A - B) < \varepsilon$ and $\{f_j\}$ converges uniformly to g on B.*

Proof. Define

$$C_{i,k} = \bigcup_{j=k}^{\infty} \{x \in A : |f_j(x) - g(x)| \geqq 2^{-i}\}.$$

By assumption,

$$\lim_{k \to \infty} \mu(C_{i,k}) = 0,$$

so that we can find $k(i)$ such that $\mu(C_{i,k(i)}) < \varepsilon 2^{-i}$. Now we have only to take

$$B = A - \bigcup_i C_{i,k(i)}.$$

Let $\{f_j\}$ be a sequence of (μ) measurable functions on G. We say that $\{f_j\}$ converges to f in (μ) measure on G if

$$\lim_{j\to\infty} \mu(\{x \in G : |f_j(x) - f(x)| > \varepsilon\}) = 0$$

for every $\varepsilon > 0$.

A property is said to hold (μ) almost everywhere (a.e.) on a set A if the set of all points for which the property does not hold has μ measure zero.

THEOREM 4.8. *Let $\{f_j\}$ be a sequence of measurable functions on G. If $\{f_j\}$ converges to f in measure on G, then there exists a subsequence $\{f_{j_k}\}$ which converges to f a.e. on G.*

PROOF. For each positive integer j, consider the set

$$E_{j,k} = \{x \in G : |f_j(x) - f(x)| > 1/k\}.$$

By assumption we can find j_k for which $\mu(E_{j_k,k}) < 2^{-k}$. Now define

$$E = \bigcap_{m=1}^{\infty} \left(\bigcup_{k=m}^{\infty} E_{j_k,k} \right).$$

Then we have

$$\mu(E) \leqq \sum_{k=m}^{\infty} \mu(E_{j_k,k}) < 2^{-m+1},$$

which shows that $\mu(E) = 0$. Moreover, if $x \in G - E$, then we can find m such that

$$|f_{j_k}(x) - f(x)| \leqq 1/k \qquad \text{for all } k \geqq m,$$

which implies that $\{f_{j_k}(x)\}$ converges to $f(x)$.

1.5 Lebesgue integral

Let μ be a measure on $\mathbf{R}^n$. A measurable function g is called a (μ) step function if the image $g(\mathbf{R}^n)$ of g is countable and the sum $\sum_r r\mu(g^{-1}(\{r\}))$ exists in the extended real number field $\overline{\mathbf{R}}$. For a function f on $\mathbf{R}^n$, we define an upper integral of f by setting

$$\int^* f \, d\mu = \inf \sum_r r\mu(g^{-1}(\{r\})),$$

where the infimum is taken over all step functions g for which

$$f(x) \leqq g(x) \qquad \text{on } \mathbf{R}^n.$$

Here we use the convention $\infty \times 0 = 0$. Similarly we define a lower integral of f by setting

$$\int_* f \, d\mu = \sup \sum_r r\mu(g^{-1}(\{r\})),$$

where the infimum is taken over all step functions g for which

$$f(x) \geqq g(x) \qquad \text{on } \mathbf{R}^n.$$

It is easy to see that if f is a step function on $\mathbf{R}^n$, then

$$\int^* f \, d\mu = \int_* f \, d\mu = \sum_r r\mu(f^{-1}(\{r\})).$$

We say that the integral of f exists if

$$\int_* f \, d\mu = \int^* f \, d\mu,$$

whose equal value is denoted by $\int f \, d\mu$. Further, we say that f is integrable if its integral exists and

$$\int f \, d\mu \in \mathbf{R}.$$

If $X \subseteqq \mathbf{R}^n$, then we write

$$\int_X f \, d\mu = \int f \, d\mu|_X,$$

when the latter has a meaning.

For a function f on $\mathbf{R}^n$, set

$$f^+(x) = \max \{f(x), 0\} \quad \text{and} \quad f^-(x) = -\min \{f(x), 0\};$$

then $f = f^+ - f^-$.

The following are easy.

Proposition 5.1. *Let f and g be functions on $\mathbf{R}^n$.*

(1) $\displaystyle\int_* f \, d\mu = -\int^* (-f) \, d\mu.$

(2) $\displaystyle\int_* f \, d\mu \leqq \int^* f \, d\mu.$

(3) $\displaystyle\int^* f \, d\mu < \infty$, *then $f(x) < \infty$ for almost all x and* $\displaystyle\int^* f^+ \, d\mu < \infty.$

(4) *If $f(x) \leqq g(x)$ for almost all x, then* $\int^* f\, d\mu \leqq \int^* g\, d\mu$.

(6) *If* $0 < c < \infty$, *then* $\int^* (cf)\, d\mu = c \int^* f\, d\mu$.

(7) *If* $\int^* f\, d\mu + \int^* g\, d\mu < \infty$, *then*

$$\int^* (f+g)\, d\mu \leqq \int^* f\, d\mu + \int^* g\, d\mu < \infty.$$

PROPOSITION 5.2. *Let $A \subseteqq \mathbf{R}^n$. If A is measurable, then*

$$\int \chi_A\, d\mu = \mu(A)$$

and if μ is regular, then

$$\int^* \chi_A\, d\mu = \mu(A).$$

PROOF. First note that

$$\int_* \chi_A\, d\mu \leqq \mu(A) \leqq \int^* \chi_A\, d\mu.$$

Hence, if A is measurable, then they are all equal.

PROPOSITION 5.3. (1) *If f is a step function, then*

$$\int f\, d\mu = \sum_r r\mu(f^{-1}(\{r\})).$$

(2) *If f is a nonnegative measurable function, then*

$$\int_* f\, d\mu = \int^* f\, d\mu \geqq 0.$$

(3) *A function f on $\mathbf{R}^n$ is integrable if and only if both f^+ and f^- are integrable; in either case,*

$$\int f\, d\mu = \int f^+\, d\mu - \int f^-\, d\mu$$

and

$$\int |f|\, d\mu = \int f^+\, d\mu + \int f^-\, d\mu.$$

PROOF. To show (2), for $1 < t < \infty$ and $j \in \mathbf{Z}$, setting

$$E_j(t) = \{x : t^j \leqq f(x) < t^{j+1}\},$$

we consider the function

$$g = \sum_j t^j \chi_{E_j(t)}.$$

Noting that g is a step function and

$$g \leqq f \leqq tg,$$

we have

$$\int^* f \, d\mu \leqq t \int g \, d\mu \leqq t \int_* f \, d\mu,$$

which gives

$$\int^* f \, d\mu = \int_* f \, d\mu,$$

as required.

THEOREM 5.1 (Fatou's lemma). *If $\{f_j\}$ is a sequence of nonnegative measurable functions on $\mathbf{R}^n$, then*

$$\liminf_{j\to\infty} \int f_j \, d\mu \geqq \int \liminf_{j\to\infty} f_j \, d\mu.$$

PROOF. Let g be a step function for which

$$\liminf_{j\to\infty} f_j \geqq g \qquad \text{on } \mathbf{R}^n.$$

Let $\{r : g^{-1}(r) \neq \emptyset\} = \{r_j\}$ and $A_j = g^{-1}(r_j)$. We may assume that $r_j > 0$ for all j. For $0 < t < 1$, consider the sets

$$B_{i,j} = \{x \in A_i : f_j(x) > tr_i\}.$$

Then it follows that

$$A_i = \bigcup_k \left(\bigcap_{j \geqq k} B_{i,j} \right),$$

so that

$$\int f_k \, d\mu \geqq \sum_i tr_i \mu(\bigcap_{j\geqq k} B_{i,j}).$$

Hence, letting $k \to \infty$, we find

$$\liminf_{j\to\infty} \int f_j \, d\mu \geqq \sum_i tr_i \mu(A_i) = t \int g \, d\mu,$$

which yields the required inequality.

THEOREM 5.2 (Lebesgue monotone convergence theorem). *If $\{f_j\}$ is a sequence of nonnegative measurable functions which converges increasingly to f a.e. on $\mathbf{R}^n$, then*

$$\lim_{j\to\infty} \int f_j \, d\mu = \int f \, d\mu.$$

PROOF. Since $\int f_j \, d\mu \leqq \int f \, d\mu$, Fatou's lemma yields the required equality.

THEOREM 5.3 (Lebesgue dominated convergence theorem). *Let $\{f_j\}$ be a sequence of measurable functions which converges to f a.e. on $\mathbf{R}^n$. If there exists an integrable function F such that $|f_j| \leqq F$ a.e. on $\mathbf{R}^n$ for every j, then*

$$\lim_{j\to\infty} \int f_j \, d\mu = \int f \, d\mu.$$

PROOF. Since $F - f_j \geqq 0$, Fatou's lemma gives

$$\liminf_{j\to\infty} \int (F - f_j) \, d\mu \geqq \int \liminf_{j\to\infty} (F - f_j) \, d\mu,$$

so that

$$\limsup_{j\to\infty} \int f_j \, d\mu \leqq \int f \, d\mu.$$

Similarly, since $F + f_j \geqq 0$,

$$\liminf_{j\to\infty} \int (F + f_j) \, d\mu \geqq \int \liminf_{j\to\infty} (F + f_j) \, d\mu,$$

so that

$$\liminf_{j\to\infty} \int f_j \, d\mu \geqq \int f \, d\mu.$$

Thus the required equality follows.

THEOREM 5.4 (Jensen's inequality). *If $\mu(\mathbf{R}^n) = 1$ and Φ is a nonnegative convex function on $\overline{\mathbf{R}}$, then*

$$\Phi(\int f d\mu) \leqq \int \Phi \circ f \, d\mu$$

for every nonnegative measurable function f on $\mathbf{R}^n$.

PROOF. If $f(\mathbf{R}^n)$ is finite, say, $\{r_1, ..., r_m\}$, then

$$\Phi(\int f d\mu) = \Phi(\sum_j r_j \mu(f^{-1}(\{r_j\}))) \leqq \sum_j \Phi(r_j)\mu(f^{-1}(\{r_j\})) = \int \Phi \circ f \, d\mu.$$

Thus the required inequality holds for step functions, and then for the general function f.

1.6 Linear functionals

Let $\mathcal{L}$ be a family of finite-valued functions on $\mathbf{R}^n$. We say that $\mathcal{L}$ is a lattice of functions on $\mathbf{R}^n$ if the following hold.

(L1) If $f \in \mathcal{L}$ and $g \in \mathcal{L}$, then $f + g \in \mathcal{L}$ and $f \wedge g \in \mathcal{L}$.

(L2) If $c \geqq 0$ and $f \in \mathcal{L}$, then $cf \in \mathcal{L}$ and $f \wedge c \in \mathcal{L}$.

(L3) If $f \in \mathcal{L}$, $g \in \mathcal{L}$ and $g \geqq f$, then $g - f \in \mathcal{L}$.

Here $f \wedge g(x) = \min\{f(x), g(x)\}$ and $g \geqq f$ means that $g(x) \geqq f(x)$ for all $x \in \mathbf{R}^n$.

If $\mathcal{L}$ is a lattice of functions on $\mathbf{R}^n$ and $f \in \mathcal{L}$, then $f^+ \in \mathcal{L}$ and $f^- \in \mathcal{L}$. Moreover, if $\mathcal{L}$ is a lattice of functions on $\mathbf{R}^n$, then so is

$$\mathcal{L}^+ = \{f \in \mathcal{L} : f \geqq 0\}.$$

THEOREM 6.1. *Let $\mathcal{L}$ be a lattice of functions on $\mathbf{R}^n$. Let λ be a function on $\mathcal{L}$ with finite value such that*

(i) *if $f \in \mathcal{L}$ and $g \in \mathcal{L}$, then $\lambda(f + g) = \lambda(f) + \lambda(g)$;*

(ii) *if $f \in \mathcal{L}$ and $c \geqq 0$, then $\lambda(cf) = c\lambda(f)$;*

(iii) *if $f \in \mathcal{L}$, $g \in \mathcal{L}$ and $g \geqq f$, then $\lambda(g) \geqq \lambda(f)$;*

(iv) *if $\{f_j\} \subseteqq \mathcal{L}$ increases to $g \in \mathcal{L}$, then $\lambda(f_j)$ increases to $\lambda(g)$.*

Then there exists a measure μ on $\mathbf{R}^n$ such that

$$\lambda(f) = \int f \, d\mu \qquad \text{whenever } f \in \mathcal{L}. \tag{6.1}$$

PROOF. First note that for $f \in \mathcal{L}^+$,

$$\lambda(f) \geqq \lambda(0 \cdot f) = 0 \cdot \lambda(f) = 0.$$

For $A \subseteqq \mathbf{R}^n$, consider the quantity

$$\mu(A) = \inf \left\{ \lim_{j\to\infty} L(f_j) \right\},$$

where the infimum is taken over all sequences $\{f_j\} \subseteqq \mathcal{L}$ such that $f_j \geqq 0$, $f_{j+1} \geqq f_j$ and $\lim_{j\to\infty} f_j(x) \geqq \chi_A(x)$. We show that μ is a measure on $\mathbf{R}^n$. For this purpose, let $A \subseteqq \bigcup_i B_i$ and take a competing sequence $\{f_{i,j}\}$ for $\mu(B_i)$. If we set

$$g_j = \sum_{i=1}^{j} f_{i,j},$$

then

$$\lim_{j\to\infty} g_j(x) \geqq \chi_A(x)$$

and

$$\lambda(g_j) = \sum_{i=1}^{j} \lambda(f_{i,j}) \leqq \sum_{i=1}^{\infty} \lim_{j\to\infty} \lambda(f_{i,j}).$$

Thus it follows that

$$\mu(A) \leqq \sum_{i=1}^{\infty} \mu(B_i).$$

Next we show that each function $f \in \mathcal{L}^+$ is measurable, and so is $f \in \mathcal{L}$. For this purpose, letting $T \subseteqq \mathbf{R}^n$ and $-\infty < a < b < \infty$, it suffices to show by Theorem 4.2 that

$$\mu(T) \geqq \mu(\{x \in T : f(x) \leqq a\}) + \mu(\{x \in T : f(x) \geqq b\}). \tag{6.2}$$

This holds trivially for $a < 0$, so that we may assume that $a \geqq 0$. Take a competing sequence $\{g_j\}$ for $\mu(T)$, and consider

$$h = \frac{f \wedge b - f \wedge a}{b - a} \qquad \text{and} \qquad k_j = h \wedge g_j.$$

Since $0 \leqq k_{j+1} - k_j \leqq g_{j+1} - g_j$, $h(x) = 1$ if $f(x) \geqq b$ and $h(x) = 0$ if $f(x) \leqq a$, we infer that

$$\lim_{j\to\infty} k_j \geqq 1 \qquad \text{on } B = \{x \in T : f(x) \geqq b\}$$

and

$$\lim_{j\to\infty} (g_j - k_j) \geqq 1 \qquad \text{on } A = \{x \in T : f(x) \leqq a\}.$$

Hence we find

$$\lim_{j\to\infty} \lambda(g_j) = \lim_{j\to\infty} [\lambda(k_j) + \lambda(g_j - k_j)] \geqq \mu(B) + \mu(A),$$

which yields (6.2).

Finally we show that

$$\lambda(f) = \int f \, d\mu \qquad \text{for } f \in \mathcal{L}^+, \tag{6.3}$$

because (6.3) gives (6.1) readily if one notes that

$$\lambda(f) = \lambda(f^+) - \lambda(f^-) = \int f^+ \, d\mu - \int f^- \, d\mu = \int f \, d\mu.$$

For this purpose we need the fact that

$$\mu(A) \geqq \lambda(h) \qquad \text{whenever } h \in \mathcal{L}^+ \text{ and } h \leqq \chi_A. \tag{6.4}$$

For $f \in \mathcal{L}$, $\varepsilon > 0$ and a positive integer j, note that

$$0 \leqq f \wedge (j\varepsilon) - f \wedge ((j-1)\varepsilon) \leqq \varepsilon \qquad \text{for all } x \in \mathbf{R}^n,$$

$$f \wedge (j\varepsilon) - f \wedge ((j-1)\varepsilon) = \varepsilon \qquad \text{whenever } f(x) \geqq j\varepsilon$$

and

$$f \wedge (j\varepsilon) - f \wedge ((j-1)\varepsilon) = 0 \qquad \text{whenever } f(x) \leqq (j-1)\varepsilon.$$

Hence it follows that

$$\begin{aligned} \lambda(f \wedge (j\varepsilon) - f \wedge ((j-1)\varepsilon)) &\geqq \varepsilon\mu(\{x : f(x) \geqq j\varepsilon\}) \\ &\geqq \int [f \wedge ((j+1)\varepsilon) - f \wedge (j\varepsilon)]\, d\mu \\ &\geqq \varepsilon\mu(\{x : f(x) \geqq (j+1)\varepsilon\}) \\ &\geqq \lambda(f \wedge ((j+2)\varepsilon) - f \wedge ((j+1)\varepsilon)) \end{aligned}$$

with the aid of (6.4), so that

$$\lambda(f \wedge (j\varepsilon)) \geqq \int [f \wedge ((j+1)\varepsilon) - f \wedge \varepsilon]\, d\mu \geqq \lambda(f \wedge ((j+2)\varepsilon) - f \wedge (2\varepsilon)).$$

By letting $j \to \infty$, we have

$$\lambda(f) \geqq \int [f - f \wedge \varepsilon]\, d\mu \geqq \lambda(f - f \wedge (2\varepsilon)),$$

which gives (6.3) by letting $\varepsilon \to 0$.

THEOREM 6.2. *Let $\mathcal{L}$ be a lattice of functions on $\mathbf{R}^n$. Let λ be a function on $\mathcal{L}$ with finite value such that*

(i) *if $f \in \mathcal{L}$ and $g \in \mathcal{L}$, then $\lambda(f+g) = \lambda(f) + \lambda(g)$;*

(ii) *if $f \in \mathcal{L}$ and $c \geqq 0$, then $\lambda(cf) = c\lambda(f)$;*

(iii) *if $f \in \mathcal{L}^+$, then $\sup\{\lambda(g) : g \in L^+ \text{ and } 0 \leqq g \leqq f\} < \infty$;*

(iv) *if $\{f_j\} \subseteqq \mathcal{L}$ increases to $g \in \mathcal{L}$, then $\lambda(f_j) \to \lambda(g)$ as $j \to \infty$.*

Set for $f \in \mathcal{L}^+$,

$$\lambda^+(f) = \sup\{\lambda(g) : g \in L^+ \text{ and } 0 \leqq g \leqq f\}$$

and

$$\lambda^-(f) = -\inf\{\lambda(g) : g \in L^+ \text{ and } 0 \leqq g \leqq f\}.$$

Then there exist measures μ^+ and μ^- on $\mathbf{R}^n$ such that

$$\lambda^+(f) = \int f\, d\mu^+, \qquad \lambda^-(f) = \int f\, d\mu^-$$

for every $f \in \mathcal{L}^+$ and

$$\lambda(f) = \int f\, d\mu^+ - \int f\, d\mu^-$$

for every $f \in \mathcal{L}$. *Moreover, if* $f \in \mathcal{L}^+$, *then there exists a sequence* $\{g_j\} \in \mathcal{L}^+$ *such that* $g_j \leqq f$ *and*

$$\lim_{j\to\infty} g_j(x) = \begin{cases} f(x) & \text{for } \mu^+\text{-a.e. } x, \\ \\ 0 & \text{for } \mu^-\text{-a.e. } x. \end{cases}$$

Proof. Let $f \in \mathcal{L}^+$ and $g \in \mathcal{L}^+$. If $g \leqq f$, then $f \geqq f - g \geqq 0$, so that

$$\lambda(g) - \lambda^-(f) \leqq \lambda(g) + \lambda(f-g) \leqq \lambda(g) + \lambda^+(f).$$

Hence it follows that

$$\lambda^+(f) - \lambda^-(f) \leqq \lambda(f) \leqq \lambda^+(f) - \lambda^-(f),$$

or

$$\lambda(f) = \lambda^+(f) - \lambda^-(f).$$

Next, if $h \in \mathcal{L}^+$ and $h \leqq f + g$, then

$$\lambda^+(f) + \lambda^+(g) \geqq \lambda(f \wedge h) + \lambda(h - f \wedge h) = \lambda(h),$$

so that

$$\lambda^+(f) + \lambda^+(g) \geqq \lambda^+(f+g).$$

Since the opposite inequality is easy, we have

$$\lambda^+(f) + \lambda^+(g) = \lambda^+(f+g).$$

Clearly, if $0 \leqq c < \infty$ and $f \geqq g$, then

$$\lambda^+(cf) = c\lambda^+(f)$$

and

$$\lambda^+(f) \geqq \lambda^+(g).$$

If $\{f_j\} \subseteqq \mathcal{L}^+$ increases to $g \in \mathcal{L}^+$, then we have for $k \in \mathcal{L}^+$ with $g \geqq k$,

$$\lambda(k) = \lim_{j\to\infty} \lambda(k \wedge f_j) \leqq \lim_{j\to\infty} \lambda^+(f_j) \leqq \lambda^+(g),$$

which gives

$$\lambda^+(g) = \lim_{j\to\infty} \lambda^+(f_j).$$

Hence we see that λ^+ satisfies all the conditions (i) - (iv) of Theorem 6.1, so that Theorem 6.1 gives μ^+ such that

$$\lambda^+(f) = \int f \, d\mu^+ \qquad \text{for every } f \in \mathcal{L}^+.$$

Similarly, there exists μ^- such that

$$\lambda^-(f) = \int f \, d\mu^- \qquad \text{for every } f \in \mathcal{L}^+.$$

For $f \in \mathcal{L}^+$, take a sequence $\{g_j\} \in \mathcal{L}^+$ such that $g_j \leqq f$ and

$$\lim_{j\to\infty} \mu(g_j) = \mu^+(f).$$

Then we see that

$$\int |f - g_j| d\mu^+ = \mu^+(f) - \mu^+(g_j) \leqq \mu^+(f) - \mu(g_j)$$

and

$$\int |g_j| d\mu^- = \mu^-(g_j) = \mu^+(g_j) - \mu(g_j) \leqq \mu^+(f) - \mu(g_j).$$

Thus $\{g_j\}$ converges to f in μ^+ measure and $\{g_j\}$ converges to 0 in μ^- measure. Hence we can choose a subsequence which converges to f μ^+-a.e. as well as to 0 μ^--a.e. .

For an open set $G \subseteqq \mathbf{R}^n$, denote by $C_0(G)$ the space of all continuous functions with compact support in G. Note that $C_0(G)$ is a lattice of functions.

Theorem 6.3 (Riesz representation theorem). *Let λ be a linear functional on $C_0(G)$ such that*

$$\sup\{\lambda(g) : 0 \leqq g \leqq f\} < \infty \qquad \textit{for any } f \in C_0(G)^+. \tag{6.5}$$

Then there exist Radon measures μ^+ and μ^- on G such that

$$\lambda(f) = \int f \, d\mu^+ - \int f \, d\mu^- \qquad \textit{whenever } f \in C_0(G). \tag{6.6}$$

Proof. We first show that the monotone convergence property (iv) of Theorem 6.2 holds. Suppose $\{f_j\} \subseteqq C_0(G)$ increases to $g \in C_0(G)$. Find a nonnegative function $h \in C_0(G)$, which is equal to 1 on a neighborhood of the support of g. For $\varepsilon > 0$, consider the compact sets

$$K_j = \{x : g(x) \geqq f_j(x) + \varepsilon\}.$$

Since $K_j \supseteqq K_{j+1}$ and $\bigcap_j K_j = \emptyset$, $K_j = \emptyset$ for large j, so that

$$0 \leqq g - f_j \leqq \varepsilon h.$$

Hence it follows that

$$|\lambda(g - f_j)| \leqq \varepsilon \max\{\lambda^+(h), \lambda^-(h)\},$$

so that

$$\lim_{j\to\infty} \lambda(f_j) = \lambda(g),$$

as required. Thus, by Theorem 6.2, there exist measures μ^+ and μ^- on G satisfying (6.5). What remains is to show that they are Radon measures on G. Let ν denote μ^+ or μ^-, and γ denote λ^+ or λ^-, respectively. Let $A \subseteqq \mathbf{R}^n$ and $\nu(A) < \infty$. Take a competing sequence $\{f_j\}$ in $C_0(G)$ for $\nu(A)$, and consider

$$V_j = \{x : f_j(x) > 1 - \varepsilon\}$$

for $0 < \varepsilon < 1$. Setting $V = \bigcup_j V_j$, we have

$$\nu(V) = \lim_{j\to\infty} \nu(V_j) \leqq \lim_{j\to\infty} (1-\varepsilon)^{-1}\gamma(f_j).$$

Hence it follows that

$$\inf\{\nu(V) : A \subseteqq V, V : \text{open}\} \leqq (1-\varepsilon)^{-1}\nu(A),$$

which proves (2.3). Similarly,

$$\nu(V) = \sup\{\nu(K) : K \subseteqq V, K : \text{compact}\}.$$

Finally we show (4.2) for $A \subseteqq \mathbf{R}^n$ and $B \subseteqq \mathbf{R}^n$ for which $\operatorname{dist}(A, B) > 0$. For this purpose, take a continuous function h on $\mathbf{R}^n$ such that $h = 1$ on A and $h = 0$ on B. If $\{f_j\}$ in $C_0(G)$ is a competing sequence for $\nu(A \cup B)$, then

$$\lim_{j\to\infty} \gamma(f_j) = \lim_{j\to\infty} [\gamma(hf_j) + \gamma((1-h)f_j)] \geqq \nu(A) + \nu(B),$$

which gives (4.2). Hence every open sets are ν measurable in view of Corollary 4.3.

Corollary 6.1. *Let λ be a linear functional on $C_0(G)$. If λ is nonnegative, that is, $\lambda(f) \geqq 0$ for every $f \in C_0(G)^+$, then there exists a Radon measure μ on G such that*

$$\lambda(f) = \int f\, d\mu \qquad \text{whenever } f \in C_0(G).$$

1.7 Riemann-Stieltjes integral

A real-valued function g on the interval $[a, b]$ is called a function of bounded variation if

$$V_a^b g = \sup \sum_{j=1}^{m} |g(t_j) - g(t_{j-1})| < \infty,$$

where the supremum is taken over all finite sequences $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$.

PROPOSITION 7.1. *If g is a function on $[a, c]$ of bounded variation, then*

$$V_a^c g = V_a^b g + V_b^c g \qquad \text{for } a < b < c. \tag{7.1}$$

For $a > b$, we define

$$V_a^b g = -V_b^a g.$$

If g is a function on $\overline{\mathbf{R}}$, then we see that (7.1) holds for every a, b and c. In addition, the function

$$v(x) = V_0^x g$$

is nondecreasing. Set

$$v(x-) = \sup_{t<x} v(t) \qquad \text{and} \qquad v(x+) = \inf_{t>x} v(t).$$

Then we have

$$v(x-) \leqq v(x) \leqq v(x+).$$

LEMMA 7.1. *If g is a function on $\overline{\mathbf{R}}$ of bounded variation, then*

$$g(x-) = \lim_{t \to x-0} g(t) \qquad \text{and} \qquad g(x+) = \lim_{t \to x+0} g(t)$$

exist. Further,

$$v(x) - v(x-) = |g(x) - g(x-)| \qquad \text{and} \qquad v(x+) - v(x) = |g(x+) - g(x)|.$$

For this purpose, it suffices to note that

$$|g(t) - g(s)| \leqq v(x-) - v(t) \qquad \text{for } t < s < x$$

and

$$|g(t) - g(s)| \leqq v(t) - v(x+) \qquad \text{for } x < s < t.$$

Let g be a function on a finite interval $[a, b]$, which is of bounded variation there. If f is a continuous function on $[a, b]$, $a = t_0 < t_1 < \cdots < t_m = b$ and $t_{j-1} \leqq s_j \leqq t_j$, then we consider the Riemann sum

$$S_g(f, \{t_j\}, \{s_j\}) = \sum_{j=1}^{m} f(s_j)[g(t_j) - g(t_{j-1})].$$

If $\delta > \max\{t_j - t_{j-1}\}$ and $\delta > \max\{t'_j - t'_{j-1}\}$, then note that

$$|S_g(f, \{t_j\}, \{s_j\}) - S_g(f, \{t'_j\}, \{s'_j\})| \leqq 2\omega(\delta) V_a^b g,$$

where $\omega(\delta) = \sup\{|f(s) - f(t)| : a \leqq s < t \leqq b, t - s < \delta\}$. Hence $S_g(f, \{t_j\}, \{s_j\})$ has a finite limit as $\max\{t_j - t_{j-1}\}$ tends to zero. The Riemann-Stieltjes integral

$$\int_a^b f \, dg$$

is defined by the limit. Clearly,

$$\left| \int_a^b f \, dg \right| \leqq \sup\{|f(x)| : a \leqq x \leqq b\} \cdot V_a^b g.$$

PROPOSITION 7.2. *If g is a function on $[a, c]$ of bounded variation, then*

$$\int_a^c f \, dg = \int_a^b f \, dg + \int_b^c f \, dg$$

for any $f \in C([a, c])$ and $a < b < c$.

For a continuous function g on $[a, b]$ of bounded variation, define a linear functional

$$\lambda(f) = \int_a^b f \, dg, \qquad f \in C([a, b]).$$

In view of the Riesz representation theorem, we have the following result.

THEOREM 7.1. *If g is a function on $\mathbf{R}$ for which $V_a^b g < \infty$ whenever $-\infty < a < b < \infty$, then there exist Radon measures μ^+ and μ^- on $\mathbf{R}$ such that*

$$\int_{-\infty}^{\infty} f \, dg = \int f \, d\mu^+ - \int f \, d\mu^- \qquad \text{for every } f \in C_0(\mathbf{R})$$

and

$$\mu^+((a, b)) + \mu^-((a, b)) \leqq V_a^b g \qquad \text{whenever } -\infty < a < b < \infty.$$

COROLLARY 7.1. *For $-\infty < a < b < \infty$,*

$$g(b+) - g(a-) = \mu^+([a, b]) - \mu^-([a, b]).$$

In Theorem 7.1, if we take $g(x) = x$, $x \in \mathbf{R}$, then μ^+ is just the one dimensional Lebesgue measure $\mathcal{L}^1$; in addition, $\mu^- = 0$. Trivially,

$$\mathcal{L}^1([a, b]) = b - a$$

and we usually write

$$\int_a^b f \, dg = \int_a^b f(x) \, dx.$$

Theorem 7.2. *If f and g are continuous functions on $[a,b]$ of bounded variation, then*

$$\int_a^b f\ dg = [f(b)g(b) - f(a)g(a)] - \int_a^b g\ df.$$

This equation, which gives a kind of reciprocity law for integrals, is known as the formula for integration by parts.

A continuous function g on $[a,b]$ of bounded variation is called absolutely continuous if

$$g(x) - g(a) = \int_a^x dg \qquad \text{for all } x \in [a,b].$$

If this is the case, then the derivative g' exists a.e., is measurable and

$$\int_a^b f\ dg = \int_a^b fg'\ dx \qquad \text{for every } f \in C([a,b]). \tag{7.2}$$

It is useful to see the following.

Corollary 7.2. *If $\varphi \in C^1(\mathbf{R})$ and μ is a Radon measure on $\mathbf{R}^n$, then*

$$\begin{aligned}\int_{B(0,a)} \varphi(|x|)d\mu &= \int_0^a \varphi(r)\ d\mu(B(0,r)) \\ &= [\varphi(r)\mu(B(0,r))]_0^a - \int_0^a \mu(B(0,r))\varphi'(r)dr\end{aligned}$$

for $a > 0$.

1.8 Product measures

Let μ and ν be measures on $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. For $C \subseteqq \mathbf{R}^{m+n}$, we define the product measure by setting

$$(\mu \times \nu)(C) = \inf \sum_{j=1}^{\infty} \mu(A_j)\nu(B_j),$$

where the infimum is taken over all sets A_j and B_j such that $C \subseteqq \bigcup_j A_j \times B_j$; here we use the convention $0 \times \infty = 0$. Clearly, $\mu \times \nu$ is a measure on $\mathbf{R}^{m+n}$.

We say that a measure μ on $\mathbf{R}^n$ is countably σ-finite if there exists $\{X_j\}$ such that $\mu(X_j) < \infty$ and $\mathbf{R}^n = \bigcup_{j=1}^{\infty} X_j$.

Theorem 8.1 (Fubini's theorem). *Let μ and ν be countably σ-finite measures on $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. Then :*

(1) *$\mu \times \nu$ is regular.*

(2) *If A is μ measurable and B is ν measurable, then $A \times B$ is $\mu \times \nu$ measurable and*

$$(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B).$$

(3) *If C is $\mu \times \nu$ measurable, then*

(3a) $C_y = \{x : (x, y) \in C\}$ *is μ measurable for ν-a.e. $y \in \mathbf{R}^n$,*

(3b) $C_x = \{y : (x, y) \in C\}$ *is ν measurable for μ-a.e. $x \in \mathbf{R}^m$ and*

$$(\mu \times \nu)(C) = \int_{\mathbf{R}^m} \nu(C_x)\, d\mu(x) = \int_{\mathbf{R}^n} \mu(C_y)\, d\nu(y).$$

(4) *If f is nonnegative and $\mu \times \nu$ measurable, then*

$$\begin{aligned} \int_{\mathbf{R}^{m+n}} f(x, y)\, d(\mu \times \nu)(x, y) &= \int_{\mathbf{R}^m} \left(\int_{\mathbf{R}^n} f(x, y)\, d\nu(y) \right) d\mu(x) \\ &= \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^m} f(x, y)\, d\mu(x) \right) d\nu(y). \end{aligned}$$

Proof. Consider the family $\mathcal{C}$ of all sets C satisfying (3a), (3b) and the repeated integrability condition

$$\int_{\mathbf{R}^m} \nu(C_x)\, d\mu(x) = \int_{\mathbf{R}^n} \mu(C_y)\, d\nu(y),$$

whose equal value is denoted by $\lambda(C)$. In view of the Lebesgue monotone convergence theorem, we see that if $\{C_j\}$ is a sequence of mutually disjoint sets in $\mathcal{C}$, then $\bigcup_j C_j \in \mathcal{C}$ and

(8.1) $$\lambda(\bigcup_j C_j) = \sum_j \lambda(C_j)$$

and if $\{C_j\}$ is a decreasing sequence in $\mathcal{C}$ and $\lambda(C_1) < \infty$, then $\bigcap_j C_j \in \mathcal{C}$ and

(8.2) $$\lambda(\bigcap_j C_j) = \lim_{j \to \infty} \lambda(C_j).$$

Further we consider the families

$$\begin{aligned} \mathcal{P}_0 &= \{A \times B : \ A \text{ is } \mu \text{ measurable and } B \text{ is } \nu \text{ measurable}\}, \\ \mathcal{P}_1 &= \left\{ \bigcup_{j=1}^{\infty} A_j \times B_j : A_j \times B_j \in \mathcal{P}_0 \right\}, \\ \mathcal{P}_2 &= \left\{ \bigcap_{j=1}^{\infty} C_j : C_j \in \mathcal{P}_1 \right\}. \end{aligned}$$

Then we insist that if $\mathcal{P}_0 \subseteqq \mathcal{C}$ and

$$\lambda(A \times B) = \mu(A) \cdot \nu(B) \tag{8.3}$$

for $A \times B \in \mathcal{P}_0$. If we note that

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$$

and

$$(A_1 \times B_1) - (A_2 \times B_2) = [(A_1 - A_2) \times B_1] \cup [(A_1 \cap A_2) \times (B_1 - B_2)],$$

we see that the finite union of sets in $\mathcal{P}_0$ is a member of $\mathcal{C}$, so that it follows from (8.1) that

$$\mathcal{P}_1 \subseteqq \mathcal{C}.$$

Moreover, since the intersection of any two sets in $\mathcal{P}_1$ belongs to $\mathcal{P}_1$, it follows from (8.2) that

$$\mathcal{P}_2 \subseteqq \mathcal{C}.$$

If V is a disjoint union of sets $A_j \times B_j \in \mathcal{P}_0$, then we have by (8.1) and (8.3)

$$\lambda(V) = \sum_j \lambda(A_j \times B_j) = \sum_j \mu(A_j)\nu(B_j).$$

Hence, if $(\mu \times \nu)(C) < \infty$, then by definition there exists a decreasing sequence $\{V_j\}$ in $\mathcal{P}_1$ such that $C \subseteqq V_j$ and

$$(\mu \times \nu)(C) = \lim_j \lambda(V_j) = \lambda(W), \qquad W = \bigcap_j V_j \in \mathcal{P}_2. \tag{8.4}$$

Hence, for $C = A \times B \in \mathcal{P}_0$, we have

$$(\mu \times \nu)(A \times B) = \lambda(W) \geqq \lambda(A \times B) = \mu(A)\nu(B) \geqq (\mu \times \nu)(A \times B),$$

so that

$$(\mu \times \nu)(A \times B) = \lambda(A \times B).$$

If $T \subseteqq \mathbf{R}^m \times \mathbf{R}^n$ and $T \subseteqq U \in \mathcal{P}_1$, then

$$\begin{aligned} &(\mu \times \nu)(T \cap (A \times B)) + (\mu \times \nu)(T - (A \times B)) \\ \leqq\ &(\mu \times \nu)(U \cap (A \times B)) + (\mu \times \nu)(U - (A \times B)) = \lambda(U), \end{aligned}$$

which implies that $A \times B$ is $\mu \times \nu$ measurable. Thus (2) is proved.

From (2) we see that every set in $\mathcal{P}_2$ is $\mu \times \nu$ measurable, and hence $\mu \times \nu$ is regular by (8.4). Thus (1) is proved.

Let C be a $\mu \times \nu$ measurable set with $(\mu \times \nu)(C) < \infty$, and take V for which $C \subseteqq V \in \mathcal{P}_2$ and

$$(\mu \times \nu)(C) = \lambda(V) = (\mu \times \nu)(V).$$

Then $(\mu \times \nu)(V - C) = 0$, so that $\lambda(W) = 0$ for some $W \in \mathcal{P}_2$ such that $V - C \subseteqq W$. Hence

$$\mu(\{x : (x, y) \in C\}) = \mu(\{x : (x, y) \in V\}) \qquad \text{for } \nu\text{-a.e. } y$$

and

$$(\mu \times \nu)(C) = \int \mu(C_y) \, d\nu(y).$$

Thus (3) follows.

Finally (4) holds for characteristic functions, in view of (3), and then, with the aid of Theorems 4.5 and 5.2, we see that (4) holds for all nonnegative measurable functions.

We define the n-dimensional Lebesgue measure by the cartesian product

$$\mathcal{L}^n = \mathcal{L}^1 \times \cdots \times \mathcal{L}^1$$

and write simply

$$\int f(x) \, d\mathcal{L}^n(x) = \int f(x) dx$$

for a nonnegative measurable function f on $\mathbf{R}^n$.

1.9 Hausdorff measures

If h is a nonnegative nondecreasing function on the interval $[0, \infty)$ satisfying the doubling condition

$$h(2r) \leqq M h(r) \qquad \text{for any } r > 0,$$

then we say that h is a measure function. We always assume that

$$h(0) = 0.$$

For $A \subseteqq \mathbf{R}^n$ and $\delta > 0$, we set

$$H_h^{(\delta)}(A) = \inf \sum_j h(r_j),$$

where the infimum is taken over all countable family $\{B_j(x_j, r_j)\}$ of balls such that $r_j < \delta$ and $A \subseteqq \bigcup_j B(x_j, r_j)$. Since $H_h^{(\delta)}(A)$ increases as δ decreases, we define the Hausdorff measure with respect to h by

$$H_h(A) = \lim_{\delta \to 0+} H_h^{(\delta)}(A).$$

Clearly, $H_h^{(\delta)}$ and H_h are measures on $\mathbf{R}^n$.

Lemma 9.1. *If* $\operatorname{dist}(A, B) > \delta > 0$, *then*

$$H_h^{(\delta)}(A \cup B) \geqq H_h^{(\delta)}(A) + H_h^{(\delta)}(B).$$

Proof. Let $\{B(x_j, r_j)\}$ be a countable family of balls such that $r_j < \delta$ and $A \cup B \subseteqq \bigcup_j B(x_j, r_j)$. Let $I' = \{j : A \cap B(x_j, r_j) \neq \emptyset\}$ and $I'' = \{j : B \cap B(x_j, r_j) \neq \emptyset\}$. Then $\{B(x_j, r_j) : j \in I'\}$ is a covering of A, so that

$$H_h^{(\delta)}(A) \leqq \sum_{j \in I'} h(r_j).$$

Similarly, $\{B(x_j, r_j) : j \in I''\}$ is a covering of B and

$$H_h^{(\delta)}(B) \leqq \sum_{j \in I''} h(r_j).$$

Since I' and I'' are disjoint, we have

$$H_h^{(\delta)}(A) + H_h^{(\delta)}(B) \leqq \sum_{j \in I} h(r_j),$$

which gives the required inequality.

By Lemma 9.1, (4.2) holds for $\mu = H_h$, so that we have the following result.

Theorem 9.1. *H_h is a Borel regular measure.*

Let σ_n be the n-dimensional measure of the unit ball $\mathbf{B} = B(0, 1)$:

$$\sigma_n = \mathcal{L}^n(\mathbf{B}).$$

Then $\mathcal{L}^n(B(0, r)) = \sigma_n r^n$. Hence we have by Fubini's theorem

$$\begin{aligned} \sigma_n &= \int_{-1}^{1} \mathcal{L}^{n-1}(\{(x_2, ..., x_n) : {x_2}^2 + \cdots + {x_n}^2 < 1 - {x_1}^2\})\, dx_1 \\ &= \sigma_{n-1} \int_{-1}^{1} (1 - {x_1}^2)^{n-1}\, dx_1 = \sigma_{n-1} \frac{\Gamma((n+1)/2)\Gamma(1/2)}{\Gamma((n+2)/2)}, \end{aligned}$$

so that

$$\sigma_n = \frac{\Gamma(1/2)^n}{\Gamma((n/2) + 1)} = \frac{\pi^{n/2}}{(n/2)\Gamma(n/2)}.$$

Now we define the m-dimensional Hausdorff measure by $\mathcal{H}^m = H_h$ with $h(r) = \sigma_m r^m$. Clearly,

$$\mathcal{H}^1 = \mathcal{L}^1.$$

For a set E and a measure function h, define the Hausdorff content :

$$M_h(E) = \inf \sum h(r_j),$$

where the infimum is taken over all coverings $\{B(x_j, r_j)\}$ of E. We see readily that $M_h(E) = 0$ if and only if $H_h(E) = 0$. Let $\mathcal{G}_j$ be the family of all cubes

$$\{x = (x_1, ..., x_n) : q_i/2^j \leqq x_i < (q_i + 1)/2^j \quad i = 1, 2, ..., n\}$$

with integers q_i, and

$$\mathcal{G} = \bigcup_j \mathcal{G}_j;$$

denote by $\ell(Q)$ the side length of Q. Define

$$m_h(E) = \inf \sum h(\ell(Q_j)),$$

where the infimum is taken over all coverings $\{Q_j\} \subseteqq \mathcal{G}$ of E. We see that

$$A^{-1}M_h(E) \leqq m_h(E) \leqq AM_h(E).$$

We say that a sequence $\{\mu_j\}$ of measures is convergent to μ vaguely on G if

$$\int f(y)d\mu(x) = \lim_{j\to\infty} \int f(y)d\mu_j(x) \qquad \text{for any } f \in C_0(G) \ ,$$

where $C_0(G)$ denotes the family of all continuous functions on $\mathbf{R}^n$ with compact support in G.

THEOREM 9.2. *Let $\{\mu_j\}$ be a sequence of measures on G. If $\{\mu_j(G)\}$ is bounded, then there exists a subsequence $\{\mu_{j_k}\}$ which converges to a measure μ vaguely on G.*

PROOF. Let $\{f_j\}$ be a countable dense family of $C_0(G)$. Since $\{\mu(f_1)\}$ is bounded, we can find a subsequence $\{\mu_{j,1}\}$ for which $\{\mu_{j,1}(f_1)\}$ converges to a number $\mu(f_1)$. Next, since $\{\mu_{j,1}(f_2)\}$ is bounded, we can find a subsequence $\{\mu_{j,2}\} \subseteqq \{\mu_{j,1}\}$ for which $\{\mu_{j,2}(f_2)\}$ converges to a number $\mu(f_2)$. Repeating this process, we take a family $\{\mu_{j,k}\}$ such that

$$\{\mu_j\} \supseteqq \{\mu_{j,1}\} \supseteqq \{\mu_{j,2}\} \supseteqq \{\mu_{j,3}\} \cdots .$$

Now we consider the diagonal sequence $\{\mu_{j,j}\}$. Then it is easy to see that $\mu_{j,j}(f_k)$ converges to $\mu(f_k)$ for each k. What remains is to show that $\mu_{j,j}(f)$ converges for every $f \in C_0(G)$. To show this, for $\varepsilon > 0$ we can find f_k for which $|f - f_k| < \varepsilon$. Then

$$|\mu_{j,j}(f) - \mu(f_k)| \leqq \varepsilon\mu_{j,j}(\mathbf{R}^n) + |\mu_{j,j}(f_k) - \mu(f_k)|,$$

from which it follows that $\{\mu_{j,j}(f)\}$ is a Cauchy sequence. Thus

$$\mu(f) = \lim_{j\to\infty} \mu_{j,j}(f)$$

gives the required measure.

THEOREM 9.3 (Frostman). *Let h be a measure function. If μ is a measure on $\mathbf{R}^n$ satisfying*

$$\mu(B(x,r)) \leqq h(r) \qquad \text{for all ball } B(x,r), \tag{9.1}$$

then

$$\mu(E) \leqq M_h(E) \qquad \text{for any set } E.$$

Conversely, if F is a compact set with $M_h(F) > 0$, then there exists a positive measure μ on F satisfying (9.1).

PROOF. Let $\{B(x_j, r_j)\}$ be a covering of a set E. Then we have

$$\mu(E) \leqq \sum_j \mu(B(x_j, r_j)) \leqq \sum_j h(r_j),$$

which gives the first assertion.

Conversely, let μ_j be a measure on F such that

$$\mu_j(Q) = h(2^{-j}) \qquad \text{whenever } Q \in \mathcal{G}_j \text{ and } Q \cap F \neq \emptyset.$$

If there exists $Q' \in \mathcal{G}_{j-1}$ such that $\mu_j(Q') > h(2^{-j+1})$, then we modify μ_j so that

$$\mu_j'(Q) = ch(2^{-j}), \qquad 0 < c = c_{Q'} < 1,$$

whenever $Q \in \mathcal{G}_j$, $Q \subseteqq Q'$ and $Q \cap F \neq \emptyset$, but

$$\mu_j'(Q') = h(2^{-j+1}).$$

We repeat this process and finally obtain μ_j^* such that

$$\mu_j^*(Q) \leqq h(2^{-k}) \qquad \text{whenever } Q \in \mathcal{G}_k,\ k \leqq j.$$

Since $\{\mu_j^*(F)\}$ is bounded, with the aid of Theorem 9.2, we can choose a subsequence which converges to μ^* vaguely. For $Q \in \mathcal{G}_j$, if we take a nonnegative continuous function which is equal to 1 on $Q \in \mathcal{G}_j$ and vanishes outside $2Q$, then we see that

$$\mu^*(Q) \leqq 3^n h(2^{-j}),$$

which proves (9.1). Let $\{Q_j\}$ be a finite family of cubes such that $\mathrm{Int}(Q_j) \cap \mathrm{Int}(Q_k) = \emptyset$ if $j \neq k$, $Q_j \cap F \neq \emptyset$ and

$$F \subseteqq \bigcup_j Q_j.$$

If m is large enough and $\bigcup_j Q_j = \bigcup_i Q_i(m)$ with $Q_i(m) \in \mathcal{G}_m$, then

$$\mu_m^*(Q_i(m)) = h(2^{-m})$$

or

$$\mu_m^*(\omega) = h(2^{-\delta}) \qquad \text{for } \omega \in \mathcal{G}_\delta.$$

Hence it follows that

$$\mu_m^*(\bigcup_j Q_j) = \sum{}' \; h(2^{-m}) + \sum{}'' \; h(2^{-\delta}) \geqq m_h(F),$$

which implies that

$$\mu^*(\mathbf{R}^n) \geqq m_h(F) > 0,$$

as required.

1.10 Maximal functions

First we show a covering lemma.

THEOREM 10.1. *Let E be a bounded set in $\mathbf{R}^n$ which is covered by the union of a family $\{B(x_j, r_j)\}$ of balls such that $\{r_j\}$ is bounded. Then there exists a disjoint subfamily $\{B(x_{j'}, r_{j'})\}$ for which*

$$\bigcup_{j'} B(x_{j'}, 5r_{j'}) \supseteqq E. \tag{10.1}$$

PROOF. For simplicity, we write $B_j = B(x_j, r_j)$ and

$$tB_j = B(x_j, tr_j), \qquad t > 0.$$

We may assume that $E \cap B_j \neq \emptyset$. Letting $A_1 = \sup\ r_j$, we choose $B_{j(1)}$ such that

$$r_{j(1)} > A_1/2.$$

Next let

$$A_2 = \sup\ \{r_j : x_j \notin 3B_{j(1)}\}.$$

If $A_2 > 0$, then we choose $B_{j(2)}$ such that

$$r_{j(2)} > A_2/2.$$

Then $r_{j(1)} > r_{j(2)}/2$ and $x_{j(2)} \notin 3B_{j(1)}$, so that $B_{j(1)} \cap B_{j(2)} = \emptyset$. Let us assume that $B_{j(1)}$,...,$B_{j(k)}$ have already chosen. If

$$A_{k+1} = \sup\ \{r_j : x_j \notin (3B_{j(1)} \cup \cdots \cup 3B_{j(k)})\} > 0,$$

then we choose $B_{j(k+1)}$ such that

$$r_{j(k+1)} > A_{k+1}/2.$$

Then, since $r_{j(\ell)} > r_{j(\ell')}/2$ when $\ell < \ell'$, we see as above that

$$B_{j(\ell)} \cap B_{j(\ell')} = \emptyset, \qquad \ell \neq \ell'.$$

Now we have obtained a (finite or infinite) sequence $\{B_{j(k)}\}$ of balls. What remains is to show that

$$\bigcup_k 5B_{j(k)} \supseteqq E. \tag{10.2}$$

Note that in case $\{B_{j(k)}\}$ is infinite, $r_{j(k)}$ tends to zero as $k \to \infty$. For any B_j, find $j(k)$ such that

$$x_j \in 3B_{j(1)} \cup \cdots \cup 3B_{j(k)}.$$

Hence it follows that $B_j \subseteqq 5B_{j(1)} \cup \cdots \cup 5B_{j(k)}$, and thus (10.2) holds.

Let μ be a Radon measure on $\mathbf{R}^n$ satisfying the doubling condition

$$\mu(2B) \leqq A\mu(B) \qquad \text{for any ball } B = B(x, r),$$

where $2B = B(x, 2r)$. A measurable function f is called locally integrable if

$$\int_K |f(x)|\, d\mu(x) < \infty$$

for every compact set K. For a locally integrable function f on $\mathbf{R}^n$, we define the maximal function

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y);$$

in case $\mu = \mathcal{L}^n$, we write Mf for $M_\mu f$.

Theorem 10.2. *For $\lambda > 0$,*

$$\mu(\{x : M_\mu f(x) > \lambda\}) \leqq \frac{A}{\lambda} \int_{\mathbf{R}^n} |f(y)|\, d\mu(y).$$

Proof. Let $E_\lambda = \{x : M_\mu f(x) > \lambda\}$. By definition, for each $x \in E_\lambda$ we can find $r(x) > 0$ such that

$$\frac{1}{\mu(B(x, r(x)))} \int_{B(x,r(x))} |f(y)|\, d\mu(y) > \lambda.$$

For $N > 0$, consider $E_{\lambda,N} = \{x \in E_\lambda : r(x) < N\}$. By Theorem 10.1, we choose a disjoint family $\{B_j\}$ of balls such that $\bigcup_j 5B_j \supseteqq E_{\lambda,N}$ and

$$\frac{1}{\mu(B_j)} \int_{B_j} |f(y)|\, d\mu(y) > \lambda.$$

Then we have

$$\begin{aligned}\mu(E_{\lambda,N}) &\leqq \sum_j \mu(5B_j) \leqq A\sum_j \mu(B_j)\\ &\leqq A\lambda^{-1}\sum_j \int_{B_j} |f(y)|\,d\mu(y)\\ &\leqq A\lambda^{-1}\int_{\mathbf{R}^n} |f(y)|\,d\mu(y),\end{aligned}$$

which yields the required inequality.

When dealing with Radon measures which fail to satisfy the doubling condition, we need to change the covering lemma by the so-called Besicovitch covering theorem.

THEOREM 10.3 (Besicovitch covering theorem). *Let E be a bounded set in $\mathbf{R}^n$. Suppose for each $x \in E$, there exists a ball $B(x, r(x))$. If $\sup_{x\in E} r(x) < \infty$, then there exists a countable subfamily $\{B(x_j, r_j)\}$ which covers E and intersects each other at most N times, where N depends only on the dimension n.*

PROOF. Let a be fixed so that $0 < a < 1$, and $A_1 = \sup_{x\in E} r(x) < \infty$. As in the proof of Theorem 10.1, we choose $B_1 = B(x_1, r(x_1))$ such that

$$r_1 > aA_1$$

and let

$$A_2 = \sup_{x\in E-B_1} r(x).$$

Next we choose $B_2 = B(x_2, r(x_2))$ such that $x_2 \in E - B_1$ and

$$r_2 > aA_2;$$

set

$$A_3 = \sup_{x\in E-(B_1\cup B_2)} r(x).$$

Repeating this process, we finally obtain $B_j = B(x_j, r(x_j))$ such that $x_j \in E - (B_1 \cup \cdots \cup B_{j-1})$ and

$$r_j > aA_j, \tag{10.3}$$

where

$$A_j = \sup_{x\in E-(B_1\cup\cdots\cup B_{j-1})} r(x).$$

By (10.3), if $\ell > j$, then

$$r_j > ar_\ell,$$

so that $\{bB_j\}$ is mutually disjoint when $0 < b < a/(a+1)$. If $\{B_j\}$ is infinite, then $r_j \to 0$ as $j \to \infty$, since E is bounded. Hence we see that $\{B_j\}$ covers E. If $y \in B_j \cap B_\ell$, $x_j \notin B_\ell$ and $x_\ell \notin B_j$, then

$$\cos \angle x_j y x_\ell = \frac{|x_j - y|^2 + |x_\ell - y|^2 - |x_j - x_\ell|^2}{2|x_j - y||x_\ell - y|} < \frac{1}{2},$$

so that the angle $\angle x_j y x_\ell$ is greater than $\pi/3$. Further, if $y \in B_j \cap B_\ell$, $x_j \in B_\ell$ and $y \notin bB_j$, then

$$\begin{aligned} \cos \angle x_j y x_\ell &\leqq \frac{|x_j - y|^2 + |x_\ell - y|^2 - {r_j}^2}{2|x_j - y||x_\ell - y|} \leqq \frac{|x_j - y|^2 + {r_\ell}^2 - {r_j}^2}{2|x_j - y| r_\ell} \\ &\leqq \frac{{r_j}^2 + {r_\ell}^2 - {r_j}^2}{2 r_j r_\ell} < \frac{1}{2a}, \end{aligned}$$

whenever $b > \sqrt{1-a^2}/a$. Now suppose $y \in B_{j_1} \cap B_{j_2} \cap \cdots \cap B_{j_m}$. If $y \in bB_{j_1} \cup bB_{j_2} \cup \cdots \cup bB_{j_m}$, then y belongs to only one ball, say, $y \in bB_{j_m}$. Then $y \in B_{j_1} \cap B_{j_2} \cap \cdots \cap B_{j_{m-1}}$ and the angles $\angle x_{j_i} y x_{j_\ell}$ are greater than $\cos^{-1}(1/2a)$, so that

$$m - 1 \leqq N$$

with a constant $N = N(a, b, n)$.

Lemma 10.1. *Let μ be a finite measure on $\mathbf{R}^n$. If $\{E_j\}$ is a disjoint family of measurable sets, then the set $\{j : \mu(E_j) > 0\}$ is at most countable.*

Proof. For each positive integer k, consider the set

$$I(k) = \{j : \mu(E_j) \geqq 1/k\}.$$

Since $\{E_j\}$ is mutually disjoint, we see that

$$\sum_{j \in I(k)} 1/k \leqq \sum_j \mu(E_j) \leqq \mu(\mathbf{R}^n) < \infty,$$

which implies that $I(k)$ is finite. Thus the required result holds if one notes that

$$\{j : \mu(E_j) > 0\} = \bigcup_{k=1}^{\infty} I(k).$$

A function f is said to be lower semicontinuous if

$$\liminf_{x \to x_0} f(x) \geqq f(x_0) \qquad \text{for all } x_0.$$

THEOREM 10.4. *If μ and ν are Radon measures on $\mathbf{R}^n$, then there exist a nonnegative μ measurable function f and a set E_∞ with $\mu(E_\infty) = 0$ such that*

$$\nu(A) = \int_A f(x)\ d\mu + \nu|_{E_\infty}(A)$$

for every measurable set A, where $\nu|_{E_\infty}(A) = \nu(E_\infty \cap A)$.

PROOF. For $t \in \mathbf{R}$, consider the hyperplane

$$\mathbf{H}_i(t) = \{(x_1, ..., x_n) : x_i = t + j2^{-m}\}.$$

By Lemma 10.1, we find an irrational number t_i such that $\mu(\mathbf{H}_i(t_i + j2^{-m})) = 0$ and $\nu(\mathbf{H}_i(t_i + j2^{-m})) = 0$ for all $m \in \mathbf{N}$ and $j \in \mathbf{Z}$. Now consider the cubes

$$I_{m,j} = \{(x_1, ..., x_n) : t_i + j_i 2^{-m} \leqq x_i < t_i + (j_i + 1)2^{-m}, \quad i = 1, ..., n\}$$

for any multi-index $j = (j_1, ..., j_n)$. For $x \in S_\mu$, define

$$\overline{F}(x) = \limsup_{m \to \infty} \frac{\nu(I_m(x))}{\mu(I_m(x))},$$

where $I_m(x) = I_{m,j}$ is chosen so that $x \in I_{m,j}$. Since $\mu(I_m(x))$ and $\nu(I_m(x))$ are lower semicontinuous by Fatou's lemma, it is seen that $\overline{F}$ is Borel measurable. We show that if $A_{\alpha,\beta} = \{x \in A : \alpha \leqq \overline{F}(x) \leqq \beta\}$, $0 < \alpha < \beta < \infty$, then

$$\alpha\mu(A_{\alpha,\beta}) \leqq \nu(A_{\alpha,\beta}) \leqq \beta\mu(A_{\alpha,\beta}). \tag{10.4}$$

Let $\varepsilon > 0$. Since ν is Borel regular, there exists an open set G such that $A_{\alpha,\beta} \subseteqq G$ and $\nu(G) < \nu(A_{\alpha,\beta}) + \varepsilon$. Then, for each $x \in A_{\alpha,\beta}$, we take the largest $I_{m,j}$ such that $x \in I_{m,j} \subseteqq G$ and $(\alpha - \varepsilon)\mu(I_{m,j}) \leqq \nu(I_{m,j})$; denote such $I_{m,j}$ by $I(x)$. If we write $\{I(x) : x \in A_{\alpha,\beta}\} = \{I_\ell\}$, then, since $\{I_\ell\}$ are mutually disjoint, we have

$$(\alpha - \varepsilon)\mu(A_{\alpha,\beta}) \leqq (\alpha - \varepsilon)\sum_\ell \mu(I_\ell) \leqq \sum_\ell \nu(I_\ell) \leqq \nu(G) < \nu(A_{\alpha,\beta}) + \varepsilon,$$

which yields the left inequality of (10.4). The right inequality of (10.4) is obtained similarly.

Now, letting $0 = r_0 < r_1 < \cdots < r_\ell < \infty$ and $A_i = \{x \in A : r_i \leqq \overline{F}(x) < r_{i+1}\}$, we see from (10.4) that

$$\sum_i r_i\mu(A_i) \leqq \sum_i \nu(A_i) \leqq \sum_i r_{i+1}\mu(A_i),$$

which gives

$$\nu(A - E_\infty) = \int_{A - E_\infty} \overline{F}(x)\ d\mu(x),$$

where $E_\infty = \{x : \overline{F}(x) = \infty\}$. On the other hand, if $K \subseteqq \{x : \overline{F}(x) \geqq i\}$, then (10.4) gives

$$\nu(K) \geqq i\mu(K) \geqq i\mu(E_\infty \cap K).$$

Hence it follows that $\mu(E_\infty) = 0$. By noting that

$$\nu(A) = \int_A \overline{F}(x) d\mu(x) + \nu(A \cap E_\infty),$$

the required assertion is now proved.

LEMMA 10.2. *Let μ be a Radon measure on $\mathbf{R}^n$ and E be a measurable subset of $\mathbf{R}^n$ with finite μ measure. Then, for any $\varepsilon > 0$, there exists a continuous function φ such that $0 \leqq \varphi \leqq 1$ on $\mathbf{R}^n$ and*

$$\int |\chi_E(x) - \varphi(x)| \, d\mu(x) < \varepsilon.$$

PROOF. Let $\varepsilon > 0$, and find a compact set K and an open set G such that $K \subseteqq E \subseteqq G$ and $\mu(E) - \varepsilon/2 < \mu(K) < \mu(G) < \mu(E) + \varepsilon/2$. Now take a continuous function φ such that $0 \leqq \varphi \leqq 1$ on $\mathbf{R}^n$, $\varphi = 1$ on K and $\varphi = 0$ outside G. Then we have $\int |\chi_E(x) - \varphi(x)| \, d\mu(x) \leqq \mu(G - K) < \varepsilon$, as required.

LEMMA 10.3. *Let μ be a Radon measure on $\mathbf{R}^n$. If f is an integrable function on $\mathbf{R}^n$ and $\varepsilon > 0$, then there exists a continuous function φ with compact support such that*

$$\int |f(x) - \varphi(x)| \, d\mu(x) < \varepsilon.$$

PROOF. For f and ε as above, take a step function $g = \sum_j a_j \chi_{E_j}$ such that

$$\int |f(x) - g(x)| \, d\mu(x) < \varepsilon/2.$$

In view of Lemma 10.2, for each j there exists $\varphi_j \in C_0(\mathbf{R}^n)$ such that

$$a_j \int |\chi_{E_j}(x) - \varphi_j(x)| \, d\mu(x) < \varepsilon 2^{-j-1}.$$

Setting $\varphi = \sum_j a_j \varphi_j$, we see that

$$\int |g(x) - \varphi(x)| \, d\mu(x) \leqq \sum_j a_j \int |\chi_{E_j}(x) - \varphi_j(x)| \, d\mu(x) < \varepsilon/2,$$

so that

$$\int |f(x) - \varphi(x)| \, d\mu(x) < \varepsilon.$$

THEOREM 10.5. *Let μ be a Radon measure on $\mathbf{R}^n$. If f is locally integrable on $\mathbf{R}^n$, then*

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\, d\mu(y) = f(x)$$

for μ almost every $x \in \mathbf{R}^n$.

PROOF. Without loss of generality, we may assume that f vanishes outside a compact set. For $r > 0$ and a measurable function g on $\mathbf{R}^n$, set

$$g_r(x) = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} g(y)\, d\mu(y)$$

and

$$A_g(x) = \limsup_{r\to 0} g_r(x) - \liminf_{r\to 0} g_r(x).$$

Our purpose is to show that $A_f(x) = 0$ for μ almost every x. If g is continuous on $\mathbf{R}^n$, then it is easy to see that $A_g(x) = 0$ for all x. Since $A_g(x) \leqq 2M_\mu g(x)$, by Theorems 10.2, we have for $\lambda > 0$,

$$\mu(\{x : A_g(x) > \lambda\}) \leqq \frac{2M}{\lambda} \int |g|\, d\mu;$$

in case μ fails to satisfy the doubling condition, we apply Theorem 10.3 instead of Theorem 10.1. If $\varphi \in C_0(\mathbf{R}^n)$, then, since $A_f(x) \leqq A_{f-\varphi}(x)$, we obtain

$$\mu(\{x : A_f(x) > \lambda\}) \leqq \frac{2M}{\lambda} \int |f - \varphi|\, d\mu.$$

Here the right-hand side can be chosen to be arbitrary small, so that it follows that $\mu(\{x : A_f(x) > \lambda\}) = 0$, which implies that $A_f = 0$ μ-a.e. on $\mathbf{R}^n$. Since $|f_r(x) - g_r(x)| \leqq M_\mu(f-g)(x)$, we see that

$$\mu(\{x : |f_r(x) - f(x)| > \lambda\}) \leqq \mu(\{x : |g_r(x) - g(x)| > \lambda/3\}) + M\lambda^{-1}\|f - g\|_1$$

for $g \in C_0(\mathbf{R}^n)$. With the aid of Lemma 10.3, it follows that $f_r \to f$ in measure, and hence Theorem 4.8 proves the conclusion.

COROLLARY 10.1. *Let μ be a Radon measure on $\mathbf{R}^n$. If f is locally integrable on $\mathbf{R}^n$, then*

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, d\mu(y) = 0 \tag{10.5}$$

for μ almost every $x \in \mathbf{R}^n$; if (10.5) *holds, then x is called a Lebesgue point.*

In fact, for each rational number c_j, we can find a set E_j such that $\mu(E_j) = 0$ and

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - c_j|\, d\mu(y) = |f(x) - c_j|$$

for every $x \in \mathbf{R}^n - E_j$. If we set $E = \bigcup_j E_j$, then $\mu(E) = 0$ and (10.5) holds for every $x \in \mathbf{R}^n - E$.

THEOREM 10.6. *If f is integrable on $\mathbf{R}^n$, then*

$$\lim_{h \to 0} \int |f(x+h) - f(x)| \, dx = 0.$$

PROOF. For $\varepsilon > 0$, by Lemma 10.3 we take $\varphi \in C_0(\mathbf{R}^n)$ for which

$$\int |f(x) - \varphi(x)| \, dx < \varepsilon.$$

Clearly,

$$\lim_{h \to 0} \int |\varphi(x+h) - \varphi(x)| \, dx = 0.$$

Hence we have

$$\begin{aligned} \int |f(x+h) - f(x)| \, dx &\leqq \int |f(x+h) - \varphi(x+h)| \, dx \\ &\quad + \int |\varphi(x+h) - \varphi(x)| \, dx + \int |\varphi(x) - f(x)| \, dx \\ &= 2\int |f(x) - \varphi(x)| \, dx + \int |\varphi(x+h) - \varphi(x)| \, dx, \end{aligned}$$

which gives

$$\limsup_{h \to 0} \int |f(x+h) - f(x)| \, dx < 2\varepsilon.$$

Thus the required equality now follows.

COROLLARY 10.2. *If f is integrable on $\mathbf{R}^n$, then*

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \, dy \to f(x) \qquad \text{in } L^1(\mathbf{R}^n) \text{ as } r \to 0.$$

1.11 Distributions

For an open set G, denote by $C_0^\infty(G)$ the family of all infinitely differentiable functions with compact support in G. For example, let

$$f(t) = \begin{cases} e^{-1/t} & \text{for } t > 0, \\ 0 & \text{for } t \leqq 0. \end{cases}$$

Since the k-th derivative $f^{(k)}(t)$ is of the form $P_k(1/t)e^{-1/t}$ for $t > 0$ with a polynomial P_k of degree $2k$, f is infinitely differentiable on the real line. Now we see that the function

$$\varphi(x) = f(1 - |x|^2)$$

is an infinitely differentiable function on $\mathbf{R}^n$ with compact support. If we choose c so that

$$c \int_{\mathbf{R}^n} \varphi(x)dx = 1,$$

then the function $\psi = c\varphi$ satisfies the following conditions :

$$\psi > 0 \quad \text{on } \mathbf{B} \qquad \text{and} \qquad \psi = 0 \quad \text{outside } \mathbf{B}. \tag{11.1}$$

$$\int_{\mathbf{R}^n} \psi \, dx = 1. \tag{11.2}$$

Such a function is called a mollifier. It is useful to consider the sequence $\{\psi_j\}$ of mollifiers such that

$$\psi_j(x) = j^n \psi(jx) \qquad j = 1, 2, ...$$

Let $\lambda = (\lambda_1, ..., \lambda_n)$ be an n-tuple of nonnegative integers; we call it a multi-index. The length of λ is defined by $|\lambda| = \lambda_1 + \cdots + \lambda_n$. We write

$$D_j = \frac{\partial}{\partial x_j} \qquad \text{and} \qquad D^\lambda = D_1^{\lambda_1} \cdots D_n^{\lambda_n} = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}}.$$

We say that a sequence $\{\varphi_j\}$ in $C_0^\infty(G)$ converges to φ in $C_0^\infty(G)$ if

(11.3) there is a compact set $K \subsetneqq G$ for which every φ_j vanishes outside K;

(11.4) for any multi-index λ, $D^\lambda \varphi_j$ converges to $D^\lambda \varphi$ uniformly.

We say that a linear functional T on $C_0^\infty(G)$ is a distribution on G if it is continuous in $C_0^\infty(G)$. A function $u \in L^1_{loc}(G)$ gives a distribution T_u by setting

$$T_u(\varphi) = \int_G \varphi u \, dx, \qquad \varphi \in C_0^\infty(G).$$

THEOREM 11.1. *If T is a distribution on G, then for any relatively compact open subset G' of G, there exist a positive constant M and a positive integer k such that*

$$|T(\varphi)| \leqq M \sum_{|\lambda| \leqq k} \sup |D^\lambda \varphi| \qquad \textit{for every } \varphi \in C_0^\infty(G').$$

PROOF. We assume that the conclusion of the theorem is not true. Then we can find $\varphi_j \in C_0^\infty(G')$ such that

$$T(\varphi_j) = 1; \tag{11.5}$$

(11.6) $$\sup |D^\lambda \varphi_j| < 1/j \qquad \text{whenever } |\lambda| < j.$$

Since (11.6) implies that $\{\varphi_j\} \to 0$ in $C_0^\infty(G)$, a contradiction follows.

We denote by $\mathcal{D}'(G)$ the family of all distributions on G, which is a linear space by the natural definitions of addition and scalar multiplication :

$$(a_1 T_1 + a_2 T_2)(\varphi) = a_1 T_1(\varphi) + a_2 T_2(\varphi), \qquad \varphi \in C_0^\infty(G),$$

for T_1, $T_2 \in \mathcal{D}'(G)$ and numbers a_1, a_2.

We now consider the differentiation of distributions. For a multi-index λ and $T \in \mathcal{D}'(G)$, we define $D^\lambda T$ by

$$(D^\lambda T)(\varphi) = (-1)^{|\lambda|} T(D^\lambda \varphi) \qquad \text{for } \varphi \in C_0^\infty(G),$$

which gives a distribution on G.

THEOREM 11.2. *If T is a distribution on G and G' is a relatively compact open subset of G, then there exist a function $f \in L^\infty(G')$ and a multi-index λ such that*

$$T = D^\lambda f \qquad \text{on } G'.$$

PROOF. By Theorem 11.1, there exist $M > 0$ and $k > 0$ such that

$$|T(\varphi)| \leqq M \sum_{|\lambda| \leqq k} \sup |D^\lambda \varphi| \qquad \text{for every } \varphi \in C_0^\infty(G').$$

By the mean value theorem, we have

$$\sup |\varphi| \leqq M \sup |(\partial/\partial x_j)\varphi|$$

for $\varphi \in C_0^\infty(G')$, so that

(11.7) $$\sup |D^\lambda \varphi| \leqq M \sup |D^{k\lambda^*} \varphi|$$

whenever $|\lambda| \leqq k$, where $\lambda^* = (1, ..., 1)$. Since

$$\varphi(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} D^{\lambda^*} \varphi(y)\, dy,$$

we have

$$\sup |\varphi| \leqq \int |D^{\lambda^*} \varphi(y)|\, dy$$

for $\varphi \in C_0^\infty(G')$, which shows by (11.7) that

$$|T(\varphi)| \leqq M \int |D^{(k+1)\lambda^*} \varphi(y)|\, dy.$$

Applying the Hahn-Banach theorem, we extend T to a bounded linear form on $L^1(G')$. Since $L^\infty(G')$ is the dual space of $L^1(G')$, we can find $f \in L^\infty(G')$ such that

$$T(\varphi) = \int [D^{(k+1)\lambda^*}\varphi(y)]f(y)\,dy$$

for $\varphi \in C_0^\infty(G')$, which implies that $T = (-1)^{(k+1)n} D^{(k+1)\lambda^*} f$, as required.

For a function $\psi \in C^\infty(G)$ and $T \in \mathcal{D}'(G)$, we define the product by

$$(\psi T)(\varphi) = T(\psi\varphi) \qquad \text{for } \varphi \in C_0^\infty(G).$$

Then the Leibniz formula remains valid.

THEOREM 11.3. *For $\psi \in C^\infty(G)$ and $T \in \mathcal{D}'(G)$,*

$$D^\lambda(\psi T) = \sum_\mu \begin{pmatrix} \lambda \\ \mu \end{pmatrix} (D^\mu \psi)(D^{\lambda-\mu}T).$$

LEMMA 11.1. *Let $\{\psi_j\}$ be a sequence of mollifiers. For any $\varphi \in C_0^\infty(G)$,*

$$\varphi * \psi_j(x) = \int \varphi(x-y)\psi_j(y)dy = \int \varphi(y)\psi_j(x-y)dy$$

converges uniformly to φ on G; in particular, $\{\varphi * \psi_j\}$ converges to φ in $C_0^\infty(G)$ as $j \to \infty$.

To show the latter assertion, it suffices to see that

$$D^\lambda(\varphi * \psi_j) = (D^\lambda \varphi) * \psi_j \qquad \text{for any multi-index } \lambda. \tag{11.8}$$

For $\psi \in C_0^\infty(\mathbf{R}^n)$ and $T \in \mathcal{D}'(\mathbf{R}^n)$, we define the convolution by setting

$$(\psi * T)(\varphi) = T(\check{\psi} * \varphi) \qquad \text{for } \varphi \in C_0^\infty(\mathbf{R}^n),$$

where $\check{\psi}(x) = \psi(-x)$. Note that if $\{\psi_j\}$ is a sequence of mollifiers, then

$$(\psi_j * T)(\varphi) \to T(\varphi) \qquad \text{as } j \to \infty \tag{11.9}$$

for every $\varphi \in C_0^\infty(\mathbf{R}^n)$, with the aid of Lemma 11.1.

THEOREM 11.4. *For $\psi \in C_0^\infty(\mathbf{R}^n)$ and $T \in \mathcal{D}'(\mathbf{R}^n)$,*

$$D^\lambda(\psi * T) = (D^\lambda \psi) * T = \psi * (D^\lambda T) \qquad \text{for any multi-index } \lambda.$$

In fact, if $\varphi \in C_0^\infty(\mathbf{R}^n)$, then

$$[D^\lambda(\psi * T)](\varphi) = (-1)^{-|\lambda|}T(\check{\psi} * D^\lambda \varphi) = T((D^\lambda\psi)\check{} * \varphi) = (D^\lambda\psi * T)(\varphi)$$

and

$$\begin{aligned}[D^\lambda(\psi * T)](\varphi) &= (-1)^{-|\lambda|}T(\check{\psi} * D^\lambda\varphi) = (-1)^{-|\lambda|}T(D^\lambda(\check{\psi} * \varphi)) \\ &= D^\lambda T(\check{\psi} * \varphi) = (\psi * D^\lambda T)(\varphi).\end{aligned}$$

THEOREM 11.5. *Let u and f be continuous functions on G. If $D_j u = f$ in the distribution sense, then $D_j u = f$ in the usual sense.*

PROOF. Take a function $\psi \in C_0^\infty(\mathbf{R}^n)$ with properties (11.1) - (11.2). For $\varepsilon > 0$, consider the mollifier $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(\varepsilon^{-1}x)$ and define

$$u_\varepsilon(x) = \int u(y)\psi_\varepsilon(x-y)\, dy$$

for $x \in G_\varepsilon = \{x \in G : \mathrm{dist}(x, \partial G) > \varepsilon\}$. Note that u_ε is infinitely differentiable on G_ε and

$$\begin{aligned}(\partial/\partial x_j)u_\varepsilon(x) &= \int u(y)[(\partial/\partial x_j)\psi_\varepsilon(x-y)]\, dy \\ &= \int u(y)[(-1)(\partial/\partial y_j)\psi_\varepsilon(x-y)]\, dy \\ &= \int f(y)\psi_\varepsilon(x-y)\, dy = f_\varepsilon(x).\end{aligned}$$

Since $u_\varepsilon \to u$ and $f_\varepsilon \to f$ locally uniformly on G, the required result now follows.

1.12 L^p-spaces

For $p \geqq 1$, denote by $L^p(\mu)$ the space of all measurable functions f for which

$$\mu_{(p)}(f) = \left(\int |f|^p\, d\mu\right)^{1/p} < \infty;$$

if $p = \infty$, then $\mu_{(\infty)}(f) = \mathrm{ess\ sup}\, |f| = \inf\{r : \mu(\{x : |f(x)| > r\}) = 0\}$.

THEOREM 12.1. *Let μ be a nonnegative Radon measure. If $1 \leqq p < \infty$, then $C_0(\mathbf{R}^n)$ is dense in $L^p(\mu)$.*

PROOF. Let E be a measurable set with $\mu(E) < \infty$. For any $\varepsilon > 0$, we can find a compact set K and an open set G such that $K \subseteqq E \subseteqq G$ and $\mu(G - K) < \varepsilon$. Now take a function $f \in C_0(G)$ such that $0 \leqq f \leqq 1$ on G and $f = 1$ on K. Then $\mu(\{x : f(x) \neq \chi_E(x)\}) \leqq \mu(G - K) < \varepsilon$, so that

$$\mu_{(p)}(f - \chi_E) \leqq [\mu(\{x : f(x) \neq \chi_E(x)\})]^{1/p} < \varepsilon^{1/p}.$$

By Minkowski's inequality given later, step functions can be approximated by functions in $C_0(\mathbf{R}^n)$, so is any function in $L^p(\mu)$; see also Lemmas 10.2 and 10.3.

THEOREM 12.2. *Let μ be a nonnegative Radon measure. If $1 \leqq p < \infty$, then $L^p(\mu)$ is separable, that is, it has a countable dense subset.*

For this purpose, we note that

$$\mathcal{A} = \left\{ \sum_j r_j \chi_{Q_j} : r_j \in \mathbf{Q}, Q_j \in \mathcal{G} \right\}$$

is a countable dense subset of $L^p(\mu)$. In fact, the characteristic function of an open set is approximated by functions in $\mathcal{A}$, so is the characteristic function of any measurable set with finite measure. It now follows that the step functions can be approximated by functions in $\mathcal{A}$.

THEOREM 12.3 (Hölder's inequality). *For $p > 0$, let $p' = p/(p-1)$. If $p \geqq 1$, $f \in L^p(\mu)$ and $g \in L^{p'}(\mu)$, then*

$$\int |fg| \, d\mu \leqq \mu_{(p)}(f) \; \mu_{(p')}(g);$$

if $0 < p < 1$, $f \in L^p(\mu)$ and $0 < \int |g|^{p'} d\mu < \infty$, then

$$\int |fg| \, d\mu \geqq \left(\int |f|^p d\mu \right)^{1/p} \left(\int |g|^{p'} d\mu \right)^{1/p'}.$$

PROOF. First let $p \geqq 1$, and note that if $\alpha > 0$, $\beta > 0$, $a > 0$, $b > 0$ and $\alpha + \beta = 1$, then $\log(\alpha a + \beta b) \geqq \alpha \log a + \beta \log b$, or $\alpha a + \beta b \geqq a^\alpha b^\beta$. Suppose $\mu_{(p)}(f) > 0$ and $\mu_{(p')}(g) > 0$, and set $F = [|f|/\mu_{(p)}(f)]^p$ and $G = [|g|/\mu_{(p')}(g)]^{p'}$. Then we have

$$[\mu_{(p)}(f)\mu_{(p')}(g)]^{-1} \int |fg| \, d\mu = \int F^{1/p} G^{1/p'} \, d\mu \leqq \int [(1/p)F + (1/p')G] \, d\mu = 1,$$

which gives the required inequality. The case $0 < p < 1$ can be shown by applying the above Hölder's inequality with $\varphi = |fg|^p$, $\psi = |g|^{-p}$ and $q = 1/p > 1$.

THEOREM 12.4 (Minkowski's inequality). *Let $f \in L^p(\mu)$ and $g \in L^p(\mu)$. In case $p \geqq 1$,*

$$\mu_{(p)}(f+g) \leqq \mu_{(p)}(f) + \mu_{(p)}(g);$$

on the contrary, in case $0 < p < 1$,

$$\mu_{(p)}(|f| + |g|) \geqq \mu_{(p)}(f) + \mu_{(p)}(g).$$

PROOF. In case $p > 1$, since $|f+g|^p \leqq |f||f+g|^{p-1} + |g||f+g|^{p-1}$, we apply Hölder's inequality to obtain

$$[\mu_{(p)}(f+g)]^p \leqq [\mu_{(p)}(f) + \mu_{(p)}(g)][\mu_{(p)}(|f+g|)]^{p-1},$$

from which the required inequality follows. The case $0 < p < 1$ can be shown similarly by use of the reverse Hölder's inequality.

In view of Minkowski's inequality, we see that $\mu_{(p)}$ is a norm in $L^p(\mu)$, which is supposed to satisfy the following properties :

(i) $\mu_{(p)}(f) \geqq 0$ for all $f \in L^p(\mu)$.

(ii) $\mu_{(p)}(f) = 0$ if and only if $f = 0$.

(iii) $\mu_{(p)}(af) = |a|\mu_{(p)}(f)$ for all $a \in \mathbf{R}$ and $f \in L^p(\mu)$.

(iv) $\mu_{(p)}(f+g) \leqq \mu_{(p)}(f) + \mu_{(p)}(g)$ for all $f \in L^p(\mu)$ and $g \in L^p(\mu)$.

THEOREM 12.5. *Let μ be a nonnegative Radon measure. For $1 \leqq p \leqq \infty$, $L^p(\mu)$ is a Banach space with the norm $\mu_{(p)}$.*

PROOF. Let $\{f_j\}$ be a Cauchy sequence in $L^p(\mu)$. Then for each positive integer k find j_k such that $\mu_{(p)}(f_j - f_m) < 2^{-k}$ whenever $j \geqq j_k$ and $m \geqq j_k$. We may assume that $j_1 < j_2 < \cdots$. Then

$$\mu_{(p)}(f_{j_{k+1}} - f_{j_k}) < 2^{-k}, \qquad k = 1, 2, \cdots.$$

Note here that $\sum_k |f_{j_{k+1}} - f_{j_k}| \in L^p(\mu)$ by Minkowski's inequality. This implies that the function

$$f_0(x) = f_{j_1}(x) + \sum_k \,[f_{j_{k+1}}(x) - f_{j_k}(x)]$$

is defined for almost every x and belongs to $L^p(\mu)$. Further,

$$\lim_{k\to\infty} \,\mu_{(p)}(f_{j_k} - f_0) = 0.$$

Since $\{f_j\}$ is a Cauchy sequence, it follows that $\lim_{j\to\infty} \,\mu_{(p)}(f_j - f_0) = 0$.

COROLLARY 12.1. *$L^2(\mu)$ is a Hilbert space with respect to the inner product*

$$(f, g) = \int \, f(x)g(x) \, d\mu(x).$$

THEOREM 12.6 (Clarkson's inequality). *Let $p \geqq 1$ and $p' = p/(p-1)$. For $f \in L^p(\mu)$ and $g \in L^p(\mu)$, if $2 \leqq p < \infty$, then*

$$\left[\mu_{(p)}\left(\frac{f+g}{2}\right)\right]^p + \left[\mu_{(p)}\left(\frac{f-g}{2}\right)\right]^p \leqq \frac{[\mu_{(p)}(f)]^p + [\mu_{(p)}(g)]^p}{2} \tag{12.1}$$

and if $1 < p < 2$, *then*

$$\left[\mu_{(p)}\left(\frac{f+g}{2}\right)\right]^{p'} + \left[\mu_{(p)}\left(\frac{f-g}{2}\right)\right]^{p'} \leqq \left(\frac{[\mu_{(p)}(f)]^p + [\mu_{(p)}(g)]^p}{2}\right)^{p'-1} \tag{12.2}$$

PROOF. First note that if $2 \leqq p < \infty$, then

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leqq \frac{|a|^p + |b|^p}{2} \qquad \text{for } a,\, b \in \mathbf{R},$$

which gives (12.1) readily. Next note that if $1 < p < 2$ and $0 < t < 1$, then

$$\begin{aligned} & (1+t)^p + (1-t)^p - 2(1+t^{p'})^{p-1} \\ = & \sum_{j=1}^{\infty} \frac{(2-p)\cdots(2j-p)}{(2j-1)!} t^{2j} \left(\frac{1-t^{(2j-p)/(p-1)}}{(2j-p)/(p-1)} - \frac{1-t^{2j/(p-1)}}{2j/(p-1)}\right) \geqq 0, \end{aligned}$$

which proves by taking $t = (1-s)/(1+s)$

$$\left(\frac{1+s}{2}\right)^{p'} + \left(\frac{1-s}{2}\right)^{p'} \leqq \left(\frac{1}{2} + \frac{s^p}{2}\right)^{1/(p-1)} \qquad \text{for } 0 < s < 1.$$

This readily yields

$$\left|\frac{a+b}{2}\right|^{p'} + \left|\frac{a-b}{2}\right|^{p'} \leqq \left(\frac{|a|^p + |b|^p}{2}\right)^{1/(p-1)} \qquad \text{for } a,\, b \in \mathbf{R}.$$

Since $p - 1 < 1$, we have

$$\begin{aligned} \left[\mu_{(p)}\left(\frac{f+g}{2}\right)\right]^{p'} + \left[\mu_{(p)}\left(\frac{f-g}{2}\right)\right]^{p'} &= \mu_{(p-1)}\left(\left|\frac{f+g}{2}\right|^{p'}\right) + \mu_{(p-1)}\left(\left|\frac{f-g}{2}\right|^{p'}\right) \\ &\leqq \left[\int \left(\left|\frac{f+g}{2}\right|^{p'} + \left|\frac{f-g}{2}\right|^{p'}\right)^{p-1} d\mu\right]^{1/(p-1)} \\ &\leqq \left(\int \frac{|f|^p + |g|^p}{2} d\mu\right)^{1/(p-1)} \\ &= \left(\frac{[\mu_{(p)}(f)]^p + [\mu_{(p)}(g)]^p}{2}\right)^{p'-1}, \end{aligned}$$

which yields (12.2).

COROLLARY 12.2. *Let* $1 < p < \infty$. *Then for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\mu_{(p)}((f+g)/2) \leqq 1 - \delta$$

whenever $\mu_{(p)}(f) \leqq 1$, $\mu_{(p)}(g) \leqq 1$ *and* $\mu_{(p)}(f-g) \geqq \varepsilon$.

THEOREM 12.7. *Let* $1 \leqq p < \infty$ *and* λ *be a linear functional on* $L^p(\mu)$ *such that* $A = \sup\{|\lambda(f)| : \mu_{(p)}(f) = 1\} < \infty$. *Then there exists a function* $k \in L^{p'}(\mu)$, $p' = p/(p-1)$, *such that*

$$\lambda(f) = \int fk \, d\mu \qquad \text{whenever } f \in L^p(\mu).$$

Moreover, if $f \in L^p(\mu)$, *then*

$$\mu_{(p)}(f) = \sup_{\{g:\mu_{(p')}(g)=1\}} \int fg \, d\mu.$$

PROOF. Consider the family $\mathcal{U}$ of all nonnegative functions $u \in L^{p'}(\mu)$ such that

$$\lambda^+(f) \geqq \int fu \, d\mu \qquad \text{whenever } f \in L^p(\mu) \text{ and } f \geqq 0.$$

Then we see that $\mu_{(p')}(u) \leqq A$, if we consider $f = (u/\mu_{(p')}(u))^{p'-1}$. If u and v are in $\mathcal{U}$, then we claim that $u \vee v = \max\{u, v\}$ are also in $\mathcal{U}$. In fact, considering the characteristic function g on the set $\{x : u(x) \geqq v(x)\}$, we have for any nonnegative function $f \in L^p(\mu)$,

$$\begin{aligned} \lambda^+(f) &= \lambda^+(fg) + \lambda^+(f(1-g)) \\ &\geqq \int fgu \, d\mu + \int (f(1-g))v \, d\mu = \int f(u \vee v) \, d\mu. \end{aligned}$$

Now we find an increasing sequence $\{u_j\}$ of functions in $\mathcal{U}$ for which

$$\lim_j \ \mu_{(p')}(u) = \sup \ \{\mu_{p'}(u) : u \in \mathcal{U}\}.$$

Clearly, $k^+ = \lim\limits_j \ u_j \in \mathcal{U}$ and $\mu_{p'}(k^+) \leqq A$. What remains is to show that

$$\lambda^+(f) = \int fk^+ \, d\mu \qquad \text{whenever } f \in L^p(\mu) \text{ and } f \geqq 0.$$

Suppose on the contrary there exists a nonnegative function $g \in L^p(\mu)$ for which

$$\lambda^+(g) > \int gk^+ \, d\mu.$$

Find a nonnegative function $h \in L^{p'}(\mu)$ such that $\{x : h(x) > 0\} = \{x : g(x) > 0\}$ and

$$\lambda^+(g) - \int gk^+ \, d\mu > \int gh \, d\mu > 0.$$

Consider

$$\tau(f) = \lambda^+(f) - \int f(k^+ + h) \, d\mu \qquad \text{for } f \in L^p(\mu).$$

By Theorem 6.2, we find a signed measure ν and a function g_0 such that $0 \leqq g_0 \leqq g$,

$$\tau(f) = \int f \, d\nu \qquad \text{for any } f \in L^p(\mu),$$

$$\tau^+(g) = \int g_0 \, d\nu^+ \qquad \text{and} \qquad \int g_0 \, d\nu^- = 0.$$

If we let χ be the characteristic function of the set $E = \{x : g_0(x) > 0\}$, then $\mu^-(E) = 0$, so that

$$\begin{aligned} \lambda^+(f) - \int f(k^+ + \chi h) \, d\mu &= \lambda^+(f(1-\chi)) - \int f(1-\chi)k^+ \, d\mu + \tau(f\chi) \\ &\geqq \tau(f\chi) = \tau^+(f\chi) \geqq 0, \end{aligned}$$

for every nonnegative function $f \in L^p(\mu)$. It follows that $k^+ + \chi h \in \mathcal{U}$, which implies that $\chi h = 0$ a.e., so that $\chi = 0$ a.e. . Thus

$$0 < \tau(g) \leqq \tau^+(g) = 0$$

and a contradiction follows.

In the same way, we find $k^- \in L^{p'}(\mu)$ such that

$$\lambda^-(f) = \int fk^- \, d\mu \qquad \text{whenever } f \in L^p(\mu).$$

Clearly, $k = k^+ - k^-$ is the required one.

For a normed space X, denote by X' the dual space of X, which consists of all bounded linear functionals on X. We say that X is reflexive if

$$X'' = (X')' = X.$$

Corollary 12.3. *If $1 < p < \infty$, then $L^p(\mu)$ is reflexive; more precisely,*

$$[L^p(\mu)]' = L^{p'}(\mu).$$

A sequence $\{f_j\}$ in X is convergent weakly to $f \in X$ if

$$\lim_{j\to\infty} T(f_j) = T(f) \qquad \text{for all } T \in X'.$$

Theorem 12.8. *Let $1 \leqq p < \infty$. If $\{f_j\} \subseteqq L^p(\mu)$ is convergent weakly to f, then*

$$\liminf_{j\to\infty} \mu_{(p)}(f_j) \geqq \mu_{(p)}(f).$$

In fact, we find

$$\int fg\, d\mu = \lim_{j\to\infty} \int f_j g\, d\mu \leqq [\liminf_{j\to\infty} \mu_{(p)}(f_j)]\mu_{(p')}(g)$$

for all $g \in L^{p'}(\mu)$.

Corollary 12.4. *Let $1 < p < \infty$. If $\{f_j\} \subseteqq L^p(\mu)$ is convergent weakly to f and $\lim_{j\to\infty} \mu_{(p)}(f_j) = \mu_{(p)}(f)$, then $\{f_j\}$ is convergent to f in $L^p(\mu)$.*

In fact, since $\liminf_{j\to\infty} \mu_{(p)}((f_j + f)/2) \geqq \mu_{(p)}(f)$ by Theorem 12.8, Minkowski's inequality yields

$$\lim_{j\to\infty} \mu_{(p)}((f_j + f)/2) = \mu_{(p)}(f).$$

Now it follows from Corollary 12.2 that

$$\lim_{j\to\infty} \mu_{(p)}(f_j - f) = 0.$$

Theorem 12.9. *Let μ be a nonnegative Radon measure, and $1 < p < \infty$. If $\{f_j\}$ is bounded in $L^p(\mu)$, then there exists a subsequence $\{f_{j_k}\}$ which converges weakly in $L^p(\mu)$.*

In fact, by Theorem 12.2, we can find a countable dense subset $\{g_k\}$ of $L^{p'}(\mu)$. Since $\{g_1(f_j)\}$ is bounded, find a subsequence $\{f_{1,j}\}$ for which $\{g_1(f_{1,j})\}$ converges. Next, find $\{f_{2,j}\} \subseteqq \{f_{1,j}\}$ for which $\{g_2(f_{2,j})\}$ converges. Repeating this process, we obtain

$$\{f_{1,j}\} \supseteqq \{f_{2,j}\} \supseteqq \cdots \supseteqq \{f_{k,j}\} \supseteqq \cdots$$

for which $\{g_k(f_{k,j})\}$ converges as $j \to \infty$. Now we may take the diagonal sequence $\{f_{j,j}\}$, which is easily seen to have the required property; see also the proof of Theorem 9.2.

The case $p = 1$ is treated in Riesz representation theorem and Theorem 9.2.

Another application of the method of diagonal sequence is the following result, which is known as the Ascoli-Arzelà theorem. A family S of continuous functions on a compact set K is said to be equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } x \in K,\ y \in K,\ |x - y| < \delta \text{ and } f \in S. \tag{12.3}$$

Theorem 12.10 (Ascoli-Arzelà). *Let S be a family of continuous functions on a compact set K such that $M = \sup_{f\in S} \|f\|_\infty < \infty$ and S is equicontinuous on K. Then there exists a sequence in S which converges uniformly on K.*

Proof. Take a countable dense set $\{x_j\}$ in K. Since $\{f(x_1) : f \in S\}$ is bounded, we can find a sequence $\{f_{1,j}\}$ for which $\{f_{1,j}(x_1)\}$ converges. Next, find $\{f_{2,j}\} \subseteqq \{f_{1,j}\}$ for which $\{f_{2,j}(x_2)\}$ converges. Repeating this process, we obtain

$$\{f_{1,j}\} \supseteqq \{f_{2,j}\} \supseteqq \cdots \supseteqq \{f_{k,j}\} \supseteqq \cdots$$

for which $\{f_{k,j}(x_k)\}$ converges as $j \to \infty$, and consider the diagonal sequence $\{f_{j,j}\}$. We have only to show that $\{f_{j,j}\}$ is a Cauchy sequence in $C(K)$. For x_k and $\varepsilon > 0$, there exists j_k such that

$$|f_{j,j}(x_k) - f_{m,m}(x_k)| < \varepsilon \qquad \text{whenever } j \geqq j_k \text{ and } m \geqq j_k.$$

By equicontinuity, find $\delta > 0$ satisfying (12.3). Then we have for $x \in K \cap B(x_k, \delta)$

$$\begin{aligned} |f_{j,j}(x) - f_{m,m}(x)| &\leqq |f_{j,j}(x) - f_{j,j}(x_k)| + |f_{j,j}(x_k) - f_{m,m}(x_k)| \\ &\quad + |f_{m,m}(x_k) - f_{m,m}(x)| < 3\varepsilon \end{aligned}$$

if $j \geqq j_k$ and $m \geqq j_k$. Since K is compact, we see that $\{f_{j,j}\}$ is a Cauchy sequence.

Chapter 2

Potentials of measures

In this chapter we are concerned with Riesz potentials of (Radon) measures; in a special case, they are called Newtonian potentials and logarithmic potentials. The semigroup property of Riesz kernels is a useful tool, in connection with energy integrals. The Riesz capacities are defined and shown to satisfy the capacitability result. The fine limit results are shown and used for the study of radial limits.

2.1 Riesz potentials

In this chapter, measures μ are all nonnegative Radon measures on $\mathbf{R}^n$. Recall that

(i) μ is countably additive, i.e.,

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

for any mutually disjoint Borel sets E_j .

(ii) $\mu(K) < \infty$ for any compact set K.

The support of μ, S_μ, is the smallest closed set F for which

$$\mu(\mathbf{R}^n - F) = 0.$$

For a set $E \subseteqq \mathbf{R}^n$, denote by $\mathcal{M}(E)$ the family of all measures with support in E; for simplicity, set $\mathcal{M} = \mathcal{M}(\mathbf{R}^n)$.

In case $0 < \alpha < n$, we define the Riesz potential $U_\alpha \mu$ of a measure μ by

$$U_\alpha \mu(x) = \int U_\alpha(x-y)d\mu(y) = \int |x-y|^{\alpha-n} d\mu(y),$$

which is sometimes called an α-potential.

We use the symbol $B(x,r)$ to denote the open ball centered at x with radius r.

THEOREM 1.1. *For any measure μ, the following assertions are equivalent:*

(1) $U_\alpha \mu \not\equiv \infty$;

(2) $\int (1+|y|)^{\alpha-n} d\mu(y) < \infty$;

(3) $\int_{\mathbf{R}^n - B(x,r)} |x-y|^{\alpha-n} d\mu(y) < \infty$ *for any* x *and any* $r > 0$;

(4) $\int_{\mathbf{R}^n - B(x,r)} |x-y|^{\alpha-n} d\mu(y) < \infty$ *for some* x *and some* $r > 0$.

Proof. First note that (1) implies (4) and

$$M^{-1}(1+|y|) < |x-y| < M(1+|y|) \qquad \text{for every } y \in \mathbf{R}^n - B(x,r). \tag{1.1}$$

Hence, (2) implies (3) and hence (4). Conversely, if (4) holds for some x_0 and $r_0 > 0$, then

$$\begin{aligned} \int (1+|y|)^{\alpha-n} d\mu(y) &= \int_{B(x_0,r_0)} (1+|y|)^{\alpha-n} d\mu(y) \\ &\quad + \int_{\mathbf{R}^n - B(x_0,r_0)} (1+|y|)^{\alpha-n} d\mu(y) \\ &\leqq \mu(B(x_0,r_0)) + M \int_{\mathbf{R}^n - B(x_0,r_0)} |x_0 - y|^{\alpha-n} d\mu(y), \end{aligned}$$

which shows (2). Thus (2), (3) and (4) are equivalent.

We show that $u(x) = \int_{B(x,1)} |x-y|^{\alpha-n} d\mu(y)$ is locally integrable. In fact, applying Fubini's theorem, we have

$$\begin{aligned} \int_{B(0,N)} u(x)\, dx &= \int_{B(0,N+1)} \left(\int_{B(y,1)} |x-y|^{\alpha-n}\, dx \right) d\mu(y) \\ &\leqq M\mu(B(0,N+1)) < \infty. \end{aligned}$$

Thus (1) follows readily from (3), and now (1) $\sim$ (4) are equivalent to each other.

A function f on an open set $G \subseteqq \mathbf{R}^n$ is said to be lower semicontinuous if

$$\liminf_{x \to x_0} f(x) \geqq f(x_0)$$

for all $x_0 \in G$; if $-f$ is lower semicontinuous, then f is called upper semicontinuous. Note that a lower semicontinuous function attains minimum on any compact subset of G. Moreover, if f is an increasing limit of a sequence of continuous functions, then we see that f is lower semicontinuous. The converse is also true.

Lemma 1.1. *Let* f *be lower semicontinuous on* G. *If* f *is nonnegative on* G, *then there exists a sequence* $\{\varphi_j\}$ *of nonnegative continuous functions on* G *which increases to* f.

PROOF. Suppose $f \not\equiv \infty$, and consider the functions

$$\varphi_j(x) = \inf\{f(y) + j|x-y| : y \in G\}.$$

Then we can show that $0 \leqq \varphi_j < \infty$, $\varphi_j \leqq \varphi_{j+1} \leqq f$ and

$$|\varphi_j(x) - \varphi_j(z)| \leqq j|x-z|$$

for all x and $z \in G$, which implies that φ_j is uniformly continuous on G. Note that

$$\varphi_j(x) \geqq \min\left\{\inf_{B(x,\delta)} f, j\delta\right\}$$

for small $\delta > 0$. Since f is lower semicontinuous at x, we also see that $\{\varphi_j(x)\}$ increases to $f(x)$.

For measures μ and ν, define the mutual α-energy by setting

$$\mathcal{E}_\alpha(\mu,\nu) = \int U_\alpha\mu(x)\, d\nu(x) = \int\int |x-y|^{\alpha-n}\, d\mu(y)d\nu(x).$$

THEOREM 1.2 (lower semicontinuity of potentials). *If $\mathcal{M}$ is the space of all measures on $\mathbf{R}^n$ with vague topology, then $\mathcal{E}_\alpha(\mu,\nu)$ is lower semicontinuous on $\mathcal{M} \times \mathcal{M}$; in particular, $U_\alpha\mu$ is lower semicontinuous on $\mathbf{R}^n$.*

PROOF. Let $\{\mu_j\}$ and $\{\nu_j\}$ be vaguely convergent to μ and ν, respectively. Take a sequence $\{k_j\}$ of nonincreasing continuous functions on $[0,\infty)$ such that $0 \leqq k_j(r) \leqq r^{\alpha-n}$ for $r > 0$ and

$$\lim_{j\to\infty} k_j(r) = r^{\alpha-n} \qquad \text{for any } r > 0.$$

Further, take a function $\varphi \in C_0^\infty(B(0,2))$ such that $0 \leqq \varphi(x) \leqq 1$ on $\mathbf{R}^n$ and $\varphi(x) = 1$ on $\mathbf{B}$; define

$$\varphi_j(x) = \varphi(j^{-1}x)$$

for $x \in \mathbf{R}^n$. If m is fixed, then

$$\int \varphi_m(x)k_m(|x-y|)\, d\mu(x) = \lim_{j\to\infty} \int \varphi_m(x)k_m(|x-y|)\, d\mu_j(x)$$

for any y. Since the convergence is locally uniform,

$$\begin{aligned} & \int\int \varphi_m(x)\varphi_m(y)k_m(|x-y|)\, d\mu(x)d\nu(y) \\ = \;& \lim_{j\to\infty} \int \varphi_m(y)\left(\int \varphi_m(x)k_m(|x-y|)\, d\mu_j(x)\right) d\nu_j(y) \\ \leqq \;& \liminf_{j\to\infty} \int\int |x-y|^{\alpha-n}\, d\mu_j(x)d\nu_j(y), \end{aligned}$$

which gives by letting $m \to \infty$,

$$\int\int |x-y|^{\alpha-n}\, d\mu(x)d\nu(y) \leqq \liminf_{j\to\infty} \int\int |x-y|^{\alpha-n}\, d\mu_j(x)d\nu_j(y).$$

If we take ν as a point measure at y, then $U_\alpha\mu(y) = \mathcal{E}_\alpha(\mu,\nu)$ is lower semicontinuous on $\mathbf{R}^n$.

THEOREM 1.3 (weak maximum principle). *For any measure μ, if $U_\alpha\mu \leqq 1$ on S_μ (the support of μ), then*

$$U_\alpha\mu \leqq 2^{n-\alpha} \qquad \textit{for all } x \in \mathbf{R}^n .$$

PROOF. For $x \in \mathbf{R}^n - S_\mu$, take $x^* \in S_\mu$ such that

$$|x^*-x| = \mathrm{dist}(x, S_\mu) = \min\{|y-x| : y \in S_\mu\}.$$

Then $|x^*-y| \leqq |x^*-x| + |x-y| \leqq 2|x-y|$ for $y \in S_\mu$, so that

$$U_\alpha\mu(x) \leqq 2^{n-\alpha}U_\alpha\mu(x^*) \leqq 2^{n-\alpha},$$

as required.

THEOREM 1.4 (continuity principle). *If $U_\alpha\mu$ is continuous as a function on S_μ, then $U_\alpha\mu$ is continuous on $\mathbf{R}^n$.*

PROOF. Since $U_\alpha\mu$ is continuous outside S_μ, we show that $U_\alpha\mu$ is continuous at $a \in S_\mu$. For any $\varepsilon > 0$, take $r > 0$ such that

$$\int_{B(a,r)} |a-y|^{\alpha-n} d\mu(y) < \varepsilon,$$

and set $\mu_r = \mu|_{B(a,r)}$ and $\nu_r = \mu|_{\mathbf{R}^n - B(a,r)}$. For x, let x' be a point in S_μ such that $|x'-x| = \mathrm{dist}(x, S_\mu)$. Since $U_\alpha\nu_r$ is continuous at a, we see that

$$\begin{aligned}
\limsup_{x\to a}\ U_\alpha\mu(x) &= \limsup_{x\to a}\ [U_\alpha\mu_r(x) + U_\alpha\nu_r(x)] \\
&\leqq 2^{n-\alpha}\limsup_{x\to a}\ U_\alpha\mu_r(x') + U_\alpha\nu_r(a) \\
&\leqq 2^{n-\alpha}\limsup_{x\to a}\ [U_\alpha\mu(x') - U_\alpha\nu_r(x')] + U_\alpha\nu_r(a) \\
&= 2^{n-\alpha}U_\alpha\mu_r(a) + U_\alpha\nu_r(a) \\
&\leqq 2^{n-\alpha}\varepsilon + U_\alpha\mu(a).
\end{aligned}$$

It thus follows that

$$\limsup_{x\to a}\ U_\alpha\mu(x) \leqq U_\alpha\mu(a).$$

Since $U_\alpha \mu$ is lower semicontinuous (see Theorem 1.2), we see that

$$\lim_{x \to a} U_\alpha \mu(x) = U_\alpha \mu(a),$$

which implies that $U_\alpha \mu$ is continuous at a.

THEOREM 1.5. *Let B be a Borel set for which $\mu(B) < \infty$. If $U_\alpha \mu$ is finite on B, then given $\varepsilon > 0$, there exists a closed set F such that $\mu(B - F) < \varepsilon$ and $U_\alpha(\mu|_F)$ is continuous on $\mathbf{R}^n$.*

PROOF. By Lusin's theorem, we can find a closed set $F \subseteqq B$ such that $\mu(B - F) < \varepsilon$ and $U_\alpha \mu$ is continuous on F as a function on F. Then

$$U_\alpha(\mu|_F) = U_\alpha \mu - U_\alpha(\mu|_{\mathbf{R}^n - F})$$

is upper semicontinuous on F as a function on F. Thus $U_\alpha(\mu|_F)$ is continuous on F as a function on F. In view of continuity principle (Theorem 1.4), it is continuous on $\mathbf{R}^n$.

COROLLARY 1.1. *Let B be a Borel set for which $\mu(B) < \infty$ and $U_\alpha \mu$ is finite on B. Then there exists a sequence $\{\mu_j\}$ such that $U_\alpha \mu_j$ is continuous and increases to $U_\alpha(\mu|_B)$.*

PROOF. By Theorem 1.5, we can take a sequence $\{F_j\}$ of closed sets such that $F_j \subseteqq B$,

$$\mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \mu(B) \tag{1.2}$$

and $U_\alpha \mu_j$ with $\mu_j = \mu|_{F_j}$ is continuous. Since we may assume that F_j is increasing, $U_\alpha \mu_j$ increases to $U_\alpha \mu|_B$ by (1.2).

THEOREM 1.6 (Riesz composition formula). *If $\alpha > 0$, $\beta > 0$ and $\alpha + \beta < n$, then there exists a positive constant $a(\alpha, \beta)$ such that*

$$\int |x - y|^{\alpha - n} |y - z|^{\beta - n} dy = a(\alpha, \beta) |x - z|^{\alpha + \beta - n}.$$

REMARK 1.1. As seen later,

$$a(\alpha, \beta) = \frac{\gamma_\alpha \gamma_\beta}{\gamma_{\alpha+\beta}}, \qquad \gamma_\alpha = \pi^{n/2 - \alpha} \frac{\Gamma(\alpha/2)}{\Gamma((n - \alpha)/2)}.$$

PROOF OF THEOREM 1.6. By change of variables, we may assume that $z = 0$. Write $x = r\xi$ and $y = s\eta$, where r, $s > 0$, $|\xi| = 1$ and $|\eta| = 1$. If θ denotes the angle between x and y, then we have

$$|x - y|^2 = r^2 + s^2 - 2rs \cos \theta,$$

so that, applying polar coordinates and change of variables : $s = rt$, we have

$$\begin{aligned}\int |x-y|^{\alpha-n}|y|^{\beta-n}dy &= \omega_{n-1}\int_0^\infty\int_0^\pi (r^2+s^2-2rs\cos\theta)^{(\alpha-n)/2}s^{\beta-n}s^{n-1}\sin^{n-2}\theta ds d\theta \\ &= \omega_{n-1}r^{\alpha+\beta-n}\int_0^\infty\int_0^\pi (1+t^2-2t\cos\theta)^{(\alpha-n)/2}t^{\beta-1}\sin^{n-2}\theta dt d\theta \\ &= a(\alpha,\beta)r^{\alpha+\beta-n} = a(\alpha,\beta)|x|^{\alpha+\beta-n},\end{aligned}$$

where ω_{n-1} denotes the surface measure of the unit sphere in $\mathbf{R}^{n-1}$.

COROLLARY 1.2. *For $\mu \in \mathcal{M}$, $\mathcal{E}_\alpha(\mu,\mu) < \infty$ if and only if $U_{\alpha/2}\mu \in L^2$, that is,*

$$\int [U_{\alpha/2}\mu(x)]^2\ dx < \infty.$$

In fact, we have

$$\begin{aligned}\int [U_{\alpha/2}\mu(x)]^2\ dx &= \int\left(\int |x-y|^{\alpha/2-n}d\mu(y)\right)\left(\int |x-z|^{\alpha/2-n}d\mu(z)\right)dx \\ &= \int\int\left(\int |x-y|^{\alpha/2-n}|x-z|^{\alpha/2-n}dx\right)d\mu(y)d\mu(z) \\ &= a(\alpha/2,\alpha/2)\int\int\ |y-z|^{\alpha-n}\ d\mu(y)d\mu(z) \\ &= a(\alpha/2,\alpha/2)\ \mathcal{E}_\alpha(\mu,\mu).\end{aligned}$$

REMARK 1.2. For a set $E \subseteqq \mathbf{R}^n$, denote by $|E|$ the n-dimensional Lebesgue measure of E. Hence, if $B = B(x,r)$, then $|B| = \sigma_n r^n$ with $\sigma_n = |B(x,1)|$. Denote by χ_E the characteristic function of E. If $|E| < \infty$, then $U_\alpha\chi_E$ is bounded and continuous on $\mathbf{R}^n$, so that $\mathcal{E}_\alpha(\chi_E,\chi_E) < \infty$; in fact, by taking a ball $B = B(x,r)$ such that $|E| = |B(x,r)|$, we see that

$$\begin{aligned}U_\alpha\chi_E(x) &= \int_{E\cap B(x,r)} |x-y|^{\alpha-n}\ dy + \int_{E-B(x,r)} |x-y|^{\alpha-n}\ dy \\ &\leqq \int_{E\cap B(x,r)} |x-y|^{\alpha-n}\ dy + r^{\alpha-n}\int_{E-B(x,r)} dy \\ &\leqq \int_{E\cap B(x,r)} |x-y|^{\alpha-n}\ dy + r^{\alpha-n}\int_{B(x,r)-E} dy \\ &\leqq \int_{E\cap B(x,r)} |x-y|^{\alpha-n}\ dy + \int_{B(x,r)-E} |x-y|^{\alpha-n}\ dy \\ &= \int_{B(x,r)} |x-y|^{\alpha-n}\ dy \\ &= \omega_n r^\alpha/\alpha\left(= e_n|E|^{\alpha/n}\right),\end{aligned}$$

where ω_n denotes the area of the unit sphere and e_n is a suitable constant. In the same way, if E is a subset of a spherical surface $S(a,r) = \partial B(a,r)$, then $U_\alpha\tilde{\chi}_E$ is bounded and continuous in case $\alpha > 1$, so that $\tilde{\chi}_E \in \mathcal{E}_\alpha$ in this case, where $d\tilde{\chi}_E = \chi_E dS$ on $S(a,r)$.

THEOREM 1.7. *Let μ, $\nu \in \mathcal{M}$. If $U_\alpha \mu \leqq U_\alpha \nu$ on $\mathbf{R}^n$, then*

$$\mu(\mathbf{R}^n) \leqq \nu(\mathbf{R}^n).$$

PROOF. For simplicity, denote by $\chi_r = \chi_{B(0,r)}$. Then

$$U_\alpha \chi_r(x) = r^\alpha U_\alpha \chi_1(x/r)$$

and

$$U_\alpha \chi_1(x) \leqq U_\alpha \chi_1(0).$$

Hence we have by Fatou's lemma

$$\begin{aligned}
U_\alpha \chi_1(0)\mu(\mathbf{R}^n) &\leqq \liminf_{r\to\infty} \int U_\alpha \chi_1(x/r)\, d\mu(x) \\
&= \liminf_{r\to\infty} r^{-\alpha} \int U_\alpha \chi_r(x)\, d\mu(x) \\
&= \liminf_{r\to\infty} r^{-\alpha} \int U_\alpha \mu(y)\, d\chi_r(y) \\
&\leqq \liminf_{r\to\infty} r^{-\alpha} \int U_\alpha \nu(y)\, d\chi_r(y) \\
&= \liminf_{r\to\infty} \int U_\alpha \chi_1(x/r)\, d\nu(x) \\
&\leqq \liminf_{r\to\infty} \int U_\alpha \chi_1(0)\, d\nu(x) \\
&= U_\alpha \chi_1(0)\nu(\mathbf{R}^n),
\end{aligned}$$

which gives the required inequality.

2.2 Fourier transform

For an integrable function f on $\mathbf{R}^n$, we define the Fourier transform $\hat{f}$ by the integral

$$\hat{f}(x) = \mathcal{F}f(x) = \int e^{-2\pi i x\cdot y} f(y)\, dy,$$

where $i = \sqrt{-1}$ and $x \cdot y$ denotes the inner product in $\mathbf{R}^n$. Similarly, define the inverse Fourier transform by

$$\mathcal{F}^* f(x) = \int e^{2\pi i x\cdot y} f(y)\, dy.$$

For $1 \leqq p < \infty$, a measurable function f on $\mathbf{R}^n$ is said to belong to $L^p(\mathbf{R}^n)$ if

$$\|f\|_p = \left(\int |f(y)|^p\, dy \right)^{1/p} < \infty;$$

in case $p = \infty$, we write $f \in L^\infty(\mathbf{R}^n)$ if f has the finite essential supremum : $\|f\|_\infty < \infty$.

THEOREM 2.1. *If $f \in L^1(\mathbf{R}^n)$, then $\hat{f}$ is uniformly continuous on $\mathbf{R}^n$ and*

$$\|\hat{f}\|_\infty \leqq \|f\|_1; \tag{2.1}$$

in fact, $\hat{f}(x)$ tends to zero as $|x| \to \infty$.

PROOF. Since $\hat{f}$ is continuous on $\mathbf{R}^n$ and (2.1) is trivially fulfilled, we show that $\hat{f}(x)$ tends to zero as $|x| \to \infty$. Let R denote a rectangle with sides parallel to the coordinate axes. If $g = \chi_R$, where χ_R denotes the characteristic function of R, then $\hat{g}$ is of the form

$$\hat{g}(x) = \prod_{j=1}^{n} \int_{a_j}^{b_j} \{\cos(2\pi x_j y_j) - i\sin(2\pi x_j y_j)\} dy_j,$$

which clearly tends to zero as $|x| \to \infty$. Now, for given $\varepsilon > 0$, find a linear combination g of such characteristic functions such that $\|f - g\|_1 < \varepsilon$. Then

$$\limsup_{|x|\to\infty} |\hat{f}(x)| \leqq \limsup_{|x|\to\infty} (\|f - g\|_1 + |\hat{g}(x)|) < \varepsilon,$$

which proves that

$$\lim_{|x|\to\infty} |\hat{f}(x)| = 0,$$

as required.

REMARK 2.1. If $f(y) = e^{-\pi\delta|y|^2}$ for $\delta > 0$, then $\hat{f}(x) = \delta^{-n/2} f(x/\delta)$.
In fact, $f \in L^1(\mathbf{R}^n)$ and

$$\begin{aligned}
\hat{f}(x) &= \int e^{-2\pi i x\cdot y} e^{-\pi\delta|y|^2}\, dy \\
&= e^{-\pi|x|^2/\delta} \int e^{-\pi\delta(y+ix/\delta)\cdot(y+ix/\delta)}\, dy \\
&= e^{-\pi|x|^2/\delta} \int e^{-\pi\delta y\cdot y}\, dy \\
&= \delta^{-n/2} e^{-\pi|x|^2/\delta}.
\end{aligned}$$

REMARK 2.2. The function $W(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ is called the Weierstrass (or Gauss-Weierstrass) kernel in $\mathbf{R}^{n+1}_+$. The following can be proved readily :

$$\int_{\mathbf{R}^n} W(x,t)dx = 1$$

and

$$\mathcal{F}(W(\cdot,t))(y) = e^{-4\pi^2 t|y|^2}$$

for $t > 0$.

There is an inequality known as Minkowski's inequality for double sequence $\{a_{ij}\}$ of nonnegative numbers; in fact, for $p \geqq 1$,

$$\left\{ \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right)^p \right\}^{1/p} \leqq \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij}^p \right)^{1/p}.$$

This inequality has an integral form as follows:

THEOREM 2.2 (Minkowski's inequality for integral). *Let μ and ν be measures on X and Y respectively. If $f(x, y)$ is a nonnegative measurable function on $X \times Y$, then*

$$\left\{ \int_X \left(\int_Y f(x, y)\, d\nu(y) \right)^p d\mu(x) \right\}^{1/p} \leqq \int_Y \left(\int_X f(x, y)^p\, d\mu(x) \right)^{1/p} d\nu(y).$$

For functions f, $g \in L^1_{loc}(\mathbf{R}^n)$, we define the convolution $h = f * g$ by

$$h(x) = f * g(x) = \int f(x - y) g(y)\, dy.$$

Clearly, $f * g = g * f$ where the above integral is absolutely convergent.

THEOREM 2.3. *Let $\varphi \in L^1(\mathbf{R}^n)$ satisfy $\int \varphi(x)\, dx = 1$, and set $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon)$. If $f \in L^p(\mathbf{R}^n)$, then $f * \varphi_\varepsilon \to f$ in $L^p(\mathbf{R}^n)$, as $\varepsilon \to 0$.*

PROOF. First we see that

$$f * \varphi_\varepsilon(x) - f(x) = \int \{f(x - \varepsilon y) - f(x)\} \varphi(y)\, dy.$$

In view of Minkowski's inequality for integral, we have

$$\|f * \varphi_\varepsilon - f\|_p \leqq \int \left(\int |f(x - \varepsilon y) - f(x)|^p\, dx \right)^{1/p} |\varphi(y)|\, dy,$$

which, together with Lebesgue's dominated convergence theorem, proves the required assertion.

THEOREM 2.4 (Young's inequality). *Let $1 \leqq p, q \leqq \infty$ and $1/r = 1/p + 1/q - 1 \geqq 0$. If $f \in L^p(\mathbf{R}^n)$ and $g \in L^q(\mathbf{R}^n)$, then*

$$\|f * g\|_r \leqq \|f\|_p \|g\|_q.$$

PROOF. Let

$$a = r, \quad \frac{1}{p} = \frac{1}{a} + \frac{1}{b}, \quad \frac{1}{q} = \frac{1}{a} + \frac{1}{c}, \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

Writing

$$|f(x-y)g(y)| = \left(|f(x-y)|^{p/a}|g(y)|^{q/a}\right)\left(|f(x-y)|^{p/b}\right)\left(|g(y)|^{q/c}\right),$$

we apply Hölder's inequality to have

$$\begin{aligned} |f*g(x)| &\leqq \int |f(x-y)g(y)|\,dy \\ &\leqq \left(\int |f(x-y)|^p|g(y)|^q\,dy\right)^{1/a}\left(\int |f(x-y)|^p\,dy\right)^{1/b}\left(\int |g(y)|^q\,dy\right)^{1/c} \\ &= \|f\|_p^{p/b}\|g\|_q^{q/c}\left(\int |f(x-y)|^p|g(y)|^q\,dy\right)^{1/a}, \end{aligned}$$

so that

$$\begin{aligned} \|f*g\|_r &\leqq \|f\|_p^{p/b}\|g\|_q^{q/c}\left\{\int\left(\int |f(x-y)|^p|g(y)|^q\,dy\right)\,dx\right\}^{1/a} \\ &= \|f\|_p^{p/b+p/a}\|g\|_q^{q/c+q/a} = \|f\|_p\|g\|_q. \end{aligned}$$

The following are easy consequences of Fubini's theorem.

THEOREM 2.5. *If $f,\, g \in L^1(\mathbf{R}^n)$, then $f*g \in L^1(\mathbf{R}^n)$ and*

$$\mathcal{F}(f*g) = \hat{f}\hat{g}.$$

In fact, we have

$$\begin{aligned} \int e^{-2\pi i x\cdot y} f*g(y)dy &= \int\left(\int e^{-2\pi i x\cdot y}f(y-z)dy\right)g(z)dz \\ &= \hat{f}(x)\int e^{-2\pi i x\cdot z}g(z)dz = \hat{f}(x)\hat{g}(x). \end{aligned}$$

THEOREM 2.6. *If $f,\, g \in L^1(\mathbf{R}^n)$, then*

$$\int \hat{f}(x)g(x)\,dx = \int f(y)\hat{g}(y)\,dy.$$

COROLLARY 2.1. *If $f,\, \hat{f} \in L^1(\mathbf{R}^n)$, then*

$$f(x) = \mathcal{F}^*(\mathcal{F}f)(x) = \mathcal{F}(\mathcal{F}^*f)(x) \qquad \textit{for almost every } x.$$

PROOF. Letting $\varphi(y) = e^{-\pi|y|^2}$, we apply Theorem 2.6 and Remark 2.1 to have

$$\int f(y)\varphi_\varepsilon(x-y)\,dy = \int \hat{f}(z)e^{2\pi i x\cdot z}\varphi(\varepsilon z)\,dz.$$

If we let $\varepsilon \to 0$, then the required result holds with the aid of Theorem 2.3.

Denote by $\mathcal{S}$ the space of all rapidly decreasing functions $\varphi \in C^\infty(\mathbf{R}^n)$; that is, $\varphi \in \mathcal{S}$ if and only if $|x|^m(\partial/\partial x)^\lambda \varphi(x)$ is bounded for any $m > 0$ and any multi-index λ. Note that

$$\mathcal{F}((\partial/\partial y)^\lambda \varphi(y))(x) = (2\pi i x)^\lambda \mathcal{F}\varphi(x) \tag{2.2}$$

and

$$\mathcal{F}((-2\pi i y)^\lambda \varphi(y))(x) = (\partial/\partial x)^\lambda \mathcal{F}\varphi(x). \tag{2.3}$$

THEOREM 2.7 (Plancherel's theorem). *If $\varphi \in \mathcal{S}$, then $\mathcal{F}\varphi \in \mathcal{S}$ and*

$$\mathcal{F}^*(\mathcal{F}\varphi) = \mathcal{F}(\mathcal{F}^*\varphi) = \varphi.$$

COROLLARY 2.2. *For $\varphi \in \mathcal{S}$, $\|\varphi\|_2 = \|\hat{\varphi}\|_2$.*

COROLLARY 2.3. *If $f \in L^2(\mathbf{R}^n)$, then there exists $g \in L^2(\mathbf{R}^n)$ such that $\|g\|_2 = \|f\|_2$ and*

$$\int f(x)\hat{\varphi}(x)\,dx = \int g(x)\varphi(x)\,dx \qquad \text{for any } \varphi \in \mathcal{S}.$$

In fact, if we take $\{\varphi_j\} \subseteqq \mathcal{S}$ which converges to f in $L^2(\mathbf{R}^n)$, then $\{\hat{\varphi}_j\}$ converges to a function g in $L^2(\mathbf{R}^n)$. In view of Corollary 2.2, it is easy to see that g has the required properties. We call g the Fourier transform of f and write $g = \hat{f}$.

THEOREM 2.8. *If $0 < \alpha < n$, then*

$$\mathcal{F}(|y|^{\alpha-n})(x) = \gamma_\alpha |x|^{-\alpha}$$

in the sense that

$$\int |y|^{\alpha-n}\hat{\varphi}(y)\,dy = \gamma_\alpha \int |x|^{-\alpha}\varphi(x)\,dx \qquad \text{for any } \varphi \in \mathcal{S} \tag{2.4}$$

with $\gamma_\alpha = \pi^{n/2-\alpha}\Gamma(\alpha/2)/\Gamma((n-\alpha)/2)$.

PROOF. In view of Remark 2.1, we have

$$\int e^{-\pi\delta|x|^2}\hat{\varphi}(x)\,dx = \delta^{-n/2}\int e^{-\pi|x|^2/\delta}\varphi(x)\,dx$$

for $\delta > 0$ and $\varphi \in \mathcal{S}$. Multiplying the equation by $\delta^{(n-\alpha)/2-1}$ and then integrating both sides with respect to δ, we obtain

$$\int \left(\int e^{-\pi\delta|x|^2}\delta^{(n-\alpha)/2-1}\,d\delta \right) \hat{\varphi}(x)\,dx = \int \left(\int e^{-\pi|x|^2/\delta}\delta^{-\alpha/2-1}d\delta \right) \varphi(x)\,dx.$$

Note here that

$$\int e^{-\pi\delta|x|^2}\delta^{(n-\alpha)/2-1}\, d\delta = (\pi|x|^2)^{-(n-\alpha)/2}\Gamma((n-\alpha)/2).$$

Hence

$$\pi^{-(n-\alpha)/2}\Gamma((n-\alpha)/2)\int |x|^{\alpha-n}\hat{\varphi}(x)\, dx = \pi^{-\alpha/2}\Gamma(\alpha/2)\int |x|^{-\alpha}\varphi(x)\, dx,$$

which leads to (2.4).

COROLLARY 2.4. *Let* $\kappa_\alpha(x) = \gamma_\alpha^{-1}|x|^{\alpha-n}$. *If* $\alpha > 0$, $\beta > 0$ *and* $\alpha+\beta < n$, *then*

$$\kappa_\alpha * \kappa_\beta = \kappa_{\alpha+\beta}$$

and

$$a(\alpha,\beta) = \frac{\gamma_\alpha\gamma_\beta}{\gamma_{\alpha+\beta}}.$$

THEOREM 2.9. *If* $\varphi \in \mathcal{S}$ *and* $0 < \alpha < n$, *then there exists* $\psi \in C^\infty$ *for which*

$$\varphi(x) = \gamma_\alpha^{-1}\int |x-y|^{\alpha-n}\psi(y)\, dy$$

and

$$\psi(x) = O(|x|^{-\alpha-n}) \qquad \text{as } |x| \to \infty. \tag{2.5}$$

In fact, let $\psi(x) = \mathcal{F}^*(|y|^\alpha\mathcal{F}\varphi(y))(x)$. If $\alpha = 2m$ with an integer m, then

$$\psi(x) = (-4\pi^2)^{-m}\Delta^m\varphi(x),$$

where Δ denotes the Laplace operator, so that (2.5) clearly holds. To derive (2.5) in the general case, we need the explicit representation of ψ.

PROOF OF THEOREM 2.9. For $\varphi \in C_0^\infty(\mathbf{R}^n)$, we see that

$$\kappa_\alpha * \varphi(x) = \gamma_\alpha^{-1}\int \varphi(x+y)|y|^{\alpha-n}\, dy$$

is analytic in $\{\alpha : 0 < \alpha < n\}$ for fixed x. Write

$$\psi(\alpha,x) = \gamma_\alpha^{-1}\int \{\varphi(x+y)-\varphi(x)\}|y|^{\alpha-n}\, dy.$$

This is well-defined for $-1 < \alpha < n$,

$$\begin{aligned}\psi(\alpha,x) &= \gamma_\alpha^{-1}\int_{\mathbf{B}} \{\varphi(x+y)-\varphi(x)\}|y|^{\alpha-n}\, dy \\ &\quad +\gamma_\alpha^{-1}\int_{\mathbf{R}^n-\mathbf{B}} \varphi(x+y)|y|^{\alpha-n}\, dy - \gamma_\alpha^{-1}\varphi(x)\left(-\frac{\omega_n}{\alpha}\right)\end{aligned}$$

and

$$\psi(\alpha, x) = O(|x|^{\alpha-n}) \qquad \text{as } |x| \to \infty. \tag{2.6}$$

Since $\lim_{\alpha\to 0} \gamma_\alpha \alpha = \omega_n$, we see that

$$\lim_{\alpha\to 0} \psi(\alpha, x) = \varphi(x).$$

Moreover, if $\alpha > 0$, then $\psi(\alpha, x) = \kappa_\alpha * \varphi(x)$. Thus $\psi(\alpha, x)$ is an analytic continuation of $\kappa_\alpha * \varphi(x)$ to the interval $(-1, 0)$. If $0 < \alpha < n - \beta < n$, then (2.6) gives

$$\kappa_\beta * \psi(\alpha, \cdot)(x) = \kappa_{\alpha+\beta} * \varphi(x),$$

so that it is true for $-1 < -\beta < \alpha < n - \beta$. In particular,

$$\kappa_\beta * \psi(-\beta, \cdot)(x) = \varphi(x)$$

for $-1 < -\beta < 0$. Thus, in this case, (2.5) follows from (2.6).

To treat the general case, we consider Taylor's expansion. Write

$$K_\ell(x, y) = \varphi(x + y) - \varphi(x) - \sum_{j=1}^{\ell} \left(\sum_{|\lambda|=j} \frac{y^\lambda}{\lambda!} \left(\frac{\partial}{\partial x}\right)^\lambda \varphi(x) \right)$$

and

$$\psi(\alpha, x) = \gamma_\alpha^{-1} \int K_\ell(x, y)|y|^{\alpha-n} dy.$$

This is well-defined and analytic in the interval $(-\ell - 1, n)$ and (2.6) holds. Now let $\ell = 2m+1$ for nonnegative integer m. One sees that for $-2m-2 < \alpha < -2m$, $\psi(\alpha, x)$ is equal to

$$\begin{aligned} \psi_m(\alpha, x) \quad &= \quad \gamma_\alpha^{-1} \int_{\mathbf{B}} \left(\varphi(x + y) - \varphi(x) - \sum_{j=1}^{m} a_j \Delta^j \varphi(x)|y|^{2j} \right) |y|^{\alpha-n} dy \\ &\quad + \gamma_\alpha^{-1} \int_{\mathbf{R}^n - \mathbf{B}} \varphi(x + y)|y|^{\alpha-n} dy - \gamma_\alpha^{-1} \varphi(x) \left(-\frac{\omega_n}{\alpha}\right) \\ &\quad - \gamma_\alpha^{-1} \sum_{j=1}^{m} a_j \Delta^j \varphi(x) \left(-\frac{\omega_n}{\alpha + 2j}\right) \end{aligned}$$

with constants $a_j = [2^j j! n(n+2)\cdots(n+2j-2)]^{-1}$, where the integral takes principal value. Since $1/[\Gamma(\alpha/2)(\alpha+2m)]$ is analytic at $\alpha = -2m$ and has value $(-1)^m m!$ there, the right-hand side is analytic in $(-2m-2, n)$ and

$$\psi_m(-2m, x) = (-4\pi^2)^{-m} \Delta^m \varphi(x);$$

moreover, in case $-2m < \alpha < -2m + 2$, it is equal to $\psi_{m-1}(\alpha, x)$. In view of (2.6), we see that $\kappa_\beta * \psi_m(\alpha, \cdot)(x)$ is well-defined for $-\beta < \alpha < n - \beta$, $2m < \beta < \min\{2m+2, n\}$, and analytic there. Since $\psi_m(\alpha, x) = \kappa_\alpha * \varphi(x)$ for $0 < \alpha < n$,

$$\kappa_\beta * \psi_m(\alpha, \cdot)(x) = \kappa_{\beta+\alpha} * \varphi(x).$$

By the analyticity of both sides, this is true for $-\beta < \alpha < n - \beta$, where $2m < \beta < 2m + 2$. In particular, letting $\alpha \to -\beta$, we have

$$\kappa_\beta * \psi_m(-\beta, \cdot)(x) = \varphi(x),$$

which was to be proved.

COROLLARY 2.5. *If* $U_\alpha \mu = U_\alpha \nu \not\equiv \infty$ *on* $\mathbf{R}^n$, *then* $\mu = \nu$.

In fact, for $\varphi \in C_0^\infty(\mathbf{R}^n)$, taking ψ such that $\varphi = U_\alpha \psi$, we have

$$\begin{aligned} \mu(\varphi) = \int U_\alpha \psi(x) d\mu(x) &= \int [U_\alpha \mu(y)] \psi(y) dy \\ &= \int [U_\alpha \nu(y)] \psi(y) dy = \nu(\varphi), \end{aligned}$$

which means that $\mu = \nu$.

2.3 Minimax lemma

Let A and B be convex subsets of a vector space. A topology is defined in B for which B is a compact Hausdorff space. Let Φ be a function on $A \times B$ with value in $\mathbf{R} \cup \{+\infty\}$.

MINIMAX LEMMA. *Suppose* Φ *satisfies*

(i) $\Phi(\cdot, b)$ *is concave on* A *for each* $b \in B$*;*

(ii) $\Phi(a, \cdot)$ *is convex on* B *for each* $a \in A$*;*

(iii) $\Phi(a, \cdot)$ *is lower semicontinuous on* B *for each* $a \in A$.

Then

$$\min_{b \in B} \sup_{a \in A} \Phi(a, b) = \sup_{a \in A} \min_{b \in B} \Phi(a, b). \tag{3.1}$$

To show this, we need the following.

LEMMA 3.1. *Let* f_j, $j = 1,\ldots, m$, *be a function on* B *with value in* $\mathbf{R} \cup \{+\infty\}$ *which are convex and lower semicontinuous. If*

$$\max_j f_j(b) > 0 \qquad \text{for all } b \in B, \tag{3.2}$$

then there exist nonnegative numbers $\lambda_1,\ldots, \lambda_m$ *such that*

$$\lambda_1 f_1(b) + \cdots + \lambda_m f_m(b) > 0 \qquad \text{for all } b \in B.$$

PROOF. First consider the case $m = 2$, and set

$$B_j = \{b \in B : f_j(b) \leqq 0\} \qquad \text{for } j = 1, 2.$$

Then each B_j is closed. If $B_1 = \emptyset$, then we may take $\lambda_1 = 1$ and $\lambda_2 = 0$, so that we may assume that both of B_j are non-empty. By (3.2),

$$f_2(b) > 0 \geqq f_1(b) \qquad \text{for } b \in B_1.$$

Noting that $-f_1/f_2$ is upper semicontinuous on B_1, we find

$$\mu_1 = \max_{b \in B_1} [-f_1(b)/f_2(b)] = -f_1(b_1)/f_2(b_1), \qquad b_1 \in B_1.$$

Similarly, we find

$$\mu_2 = \max_{b \in B_2} [-f_2(b)/f_1(b)] = -f_2(b_2)/f_1(b_2), \qquad b_2 \in B_2.$$

Now we find $\lambda > 0$ such that

$$\lambda f_1(b) + f_2(b) > 0 \qquad \text{for all } b \in B.$$

This is clearly true for $b \notin B_1 \cup B_2$. If $b \in B_1$, then

$$\lambda f_1(b) + f_2(b) \geqq (1 - \lambda\mu_1) f_2(b)$$

and if $b \in B_2$, then

$$\lambda f_1(b) + f_2(b) \geqq (\lambda - \mu_2) f_1(b).$$

Hence we have only to find $\lambda > 0$ such that

$$1 - \lambda\mu_1 > 0 \qquad \text{and} \qquad \lambda - \mu_2 > 0.$$

For this purpose, we may assume that $\mu_1 \neq 0$ and $\mu_2 \neq 0$. Our task is to show that $\mu_1\mu_2 < 1$. For $0 < t < 1$, let $b_t = tb_1 + (1-t)b_2$. Then

$$f_1(b_t) \leqq t f_1(b_1) + (1-t) f_1(b_2).$$

Since $f_1(b_1) < 0 < f_1(b_2)$, we can find t such that

$$t f_1(b_1) + (1-t) f_1(b_2) = 0,$$

so that

$$t[-\mu_1 f_2(b_1)] + (1-t)[-f_2(b_2)/\mu_2] = 0$$

or

$$t\mu_1\mu_2 f_2(b_1) + (1-t) f_2(b_2) = 0.$$

Since $f_1(b_t) \leqq 0$, $f_2(b_t) > 0$ by assumption (3.2), so that

$$tf_2(b_1) + (1-t)f_2(b_2) \geqq f_2(b_t) > 0.$$

Now it follows that

$$(1 - \mu_1\mu_2)tf_2(b_1) > 0,$$

which implies that $1 - \mu_1\mu_2 > 0$ as required. Thus the case $m = 2$ is proved.

We show the present lemma by induction on m. Assume that the case $m-1$ is true, and consider

$$B_m = \{b \in B : f_m(b) \leqq 0\}.$$

If $B_m = \emptyset$, then we may take $\lambda_1 = \cdots = \lambda_{m-1} = 0$ and $\lambda_m = 1$. Since f_m is lower semicontinuous, B_m is a compact subset of B and $\max\limits_{1 \leqq j \leqq m-1} f_j(b) > 0$ for all $b \in B_m$, by assumption on induction, there exist nonnegative numbers λ_1,..., λ_{m-1} such that

$$f(b) \equiv \lambda_1 f_1(b) + \cdots + \lambda_{m-1}f_{m-1}(b) > 0 \qquad \text{for all } b \in B_m.$$

Note here that f and f_m are convex and lower semicontinuous on B and

$$\max\{f(b), f_m(b)\} > 0 \qquad \text{for all } b \in B.$$

Hence, by considering the case $m = 2$, we find $\lambda > 0$ such that

$$\lambda f(b) + f_m(b) > 0$$

or

$$[\lambda\lambda_1]f_1(b) + \cdots + [\lambda\lambda_{m-1}]f_{m-1}(b) + f_m(b) > 0$$

for all $b \in B$, as required.

Proof of Minimax Lemma. Now we show (3.1). Since inequality "$\geqq$" is trivial, we show the converse inequality. For this purpose we may assume that the right-hand side of (3.1) is finite. By subtracting a finite constant from Φ, we may also assume that

$$\sup_{a \in A} \min_{b \in B} \Phi(a, b) = 0. \tag{3.3}$$

By assumption (iii), for each $a \in A$,

$$B_a = \{b \in B : \Phi(a, b) \leqq 0\}$$

is closed. By (3.3), B_a is non-empty. We show that $\{B_a\}$ has the finite intersection property. To show this, suppose on the contrary

$$B_{a_1} \cap \cdots \cap B_{a_m} = \emptyset$$

for some $a_1, ..., a_m \in A$. Consider $f_j(b) = \Phi(a_j, b)$. Applying Lemma 3.1 with $f_j(b) = \Phi(a_j, b)$, we find nonnegative numbers λ_1,..., λ_m such that

$$\lambda_1 \Phi(a_1, b) + \cdots + \lambda_m \Phi(a_m, b) > 0 \qquad \text{for all } b \in B.$$

If we set

$$a_0 = (\lambda_1 a_1 + \cdots \lambda_m a_m)/(\lambda_1 + \cdots + \lambda_m),$$

then (i) gives

$$\Phi(a_0, b) > 0 \qquad \text{for all } b \in B,$$

which contradicts (3.3). Consequently, $\{B_a\}$ is shown to have the intersection property. Since B is compact, there exists $b_0 \in B$ for which

$$b_0 \in \bigcap_{a \in A} B_a,$$

which implies that

$$\min_{b \in B} \sup_{a \in A} \Phi(a, b) \leqq \sup_{a \in A} \Phi(a, b_0) \leqq 0,$$

as required.

2.4 Capacity

For a set $E \subseteqq \mathbf{R}^n$, we define a capacity by

$$C_\alpha(E) = \inf \ \mu(\mathbf{R}^n),$$

where the infimum is taken over all nonnegative measures μ such that

$$U_\alpha \mu(x) \geqq 1 \qquad \text{for any } x \in E.$$

Similarly, we define a capacity by

$$c_\alpha(E) = \sup \ \nu(\mathbf{R}^n),$$

where the supremum is taken over all nonnegative measures ν with support in E such that

$$U_\alpha \nu(x) \leqq 1 \qquad \text{for any } x \in \mathbf{R}^n.$$

If μ and ν are competing measures for $C_\alpha(E)$ and $c_\alpha(E)$, respectively, then

$$\nu(E) \leqq \int U_\alpha \mu \, d\nu = \int U_\alpha \nu \, d\mu \leqq \mu(\mathbf{R}^n),$$

so that

(4.1) $$c_\alpha(E) \leqq C_\alpha(E).$$

First, we collect elementary facts about the capacities.

THEOREM 4.1. (1) *C_α is nondecreasing, that is,*

$$C_\alpha(E_1) \leqq C_\alpha(E_2) \qquad \text{if } E_1 \subseteqq E_2.$$

(2) *C_α is countably subadditive, that is,*

$$C_\alpha(\bigcup_{j=1}^{\infty} E_j) \leqq \sum_{j=1}^{\infty} C_\alpha(E_j).$$

(3) *C_α is an outer capacity, that is,*

$$C_\alpha(E) = \inf\{C_\alpha(G) : E \subseteqq G,\ G \text{ is open}\}.$$

(4) *If $C_\alpha(E) = 0$, then there exists a measure μ such that $U_\alpha \mu \not\equiv \infty$ and $U_\alpha \mu(x) = \infty$ for every $x \in E$.*

THEOREM 4.2. (1) *c_α is nondecreasing.*

(2) *c_α is countably subadditive on the Borel family $\mathcal{B}$.*

(3) *c_α is an inner capacity, that is,*

$$c_\alpha(E) = \sup\{c_\alpha(K) : K \subseteqq E,\ K \text{ is compact }\}.$$

For a proof of (2) of Theorem 4.2, it suffices to note that if E is a Borel set and $U_\alpha \mu \leqq 1$ on $\mathbf{R}^n$, then

$$\mu(E) = \sup_{K \subseteqq E, K:\text{compact}} \mu(K) \leqq c_\alpha(E).$$

By minimax lemma, we can prove

THEOREM 4.3. *For any compact set K, we have*

$$C_\alpha(K) = c_\alpha(K). \tag{4.2}$$

For this purpose, set $A = \{\mu : \mu(\mathbf{R}^n) = 1\}$, $B = \{\nu : \nu(K) = 1, S\nu \subseteqq K\}$ and

$$\Phi(\mu, \nu) = \mathcal{E}_\alpha(\mu, \nu) = \int U_\alpha \mu \, d\nu.$$

Here we consider the vague convergence topology in B. Then it suffices to note that

$$[C_\alpha(K)]^{-1} = \sup_{\mu \in A} \inf_{\nu \in B} \mathcal{E}_\alpha(\mu, \nu)$$

and

$$[c_\alpha(K)]^{-1} = \inf_{\nu \in B} \sup_{\mu \in A} \mathcal{E}_\alpha(\mu, \nu).$$

Corollary 4.1. *Let E be a Borel set with $C_\alpha(E) = 0$. If $\mathcal{E}_\alpha(\mu, \mu) < \infty$, then $\mu(E) = 0$.*

In fact, if $\mu(E) > 0$, then, in view of Theorem 1.5, we can find a compact set $K \subseteqq E$ such that $\mu(K) > 0$ and $U_\alpha(\mu|_K)$ is continuous as a function on K. Now continuity principle as well as weak maximum principle implies that $U_\alpha(\mu|_K)$ is bounded and continuous on $\mathbf{R}^n$, so that $c_\alpha(K) > 0$. Thus a contradiction follows.

Corollary 4.2. *If G is an open set, then $C_\alpha(G) = c_\alpha(G)$.*

Proof. Let $\{K_j\}$ be a sequence of compact sets such that $K_j \subseteqq \mathrm{Int}(K_{j+1})$ and $\bigcup_{j=1}^{\infty} K_j = G$. For each j, take a measure μ_j such that $\mu_j(\mathbf{R}^n) < C_\alpha(K_j) + 1/j$ and $U_\alpha \mu_j \geqq 1$ on K_j. Here we may assume that $C_\alpha(K_j) = c_\alpha(K_j)$ is bounded. Then, if necessary, by passing to a subsequence, μ_j converges vaguely to a measure μ. Let $x \in G$. Since $U_\alpha \chi_{B(x,r)}$ is bounded, continuous and further vanishes at infinity, we see that

$$\begin{aligned} \int_{B(x,r)} U_\alpha \mu \, dy &= \int U_\alpha \chi_{B(x,r)} \, d\mu = \lim_{j\to\infty} \int U_\alpha \chi_{B(x,r)} \, d\mu_j \\ &= \lim_{j\to\infty} \int_{B(x,r)} U_\alpha \mu_j \, dy \geqq |B(x,r)| \end{aligned}$$

whenever $\overline{B(x,r)} \subseteqq G$. Letting $r \to 0$, we have

$$U_\alpha \mu(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} U_\alpha \mu \, dy \geqq 1,$$

which proves by Theorem 4.3

$$C_\alpha(G) \leqq \mu(\mathbf{R}^n) \leqq \liminf_{j\to\infty} \mu_j(\mathbf{R}^n) \leqq c_\alpha(G).$$

In view of (4.1), $C_\alpha(G) \geqq c_\alpha(G)$ and then the equality holds.

A function f is said to be α-quasicontinuous on G if for any $\varepsilon > 0$, there exists an open set ω such that $C_\alpha(\omega) < \varepsilon$ and f is continuous as a function on $G - \omega$.

Theorem 4.4. *If $U_\alpha \mu \not\equiv \infty$, then $U_\alpha \mu$ is α-quasicontinuous on $\mathbf{R}^n$.*

Proof. We show that $U_\alpha \mu$ is α-quasicontinuous on any open ball B. Write

$$\begin{aligned} U_\alpha \mu(x) &= \int_B |x-y|^{\alpha-n} d\mu(y) + \int_{\mathbf{R}^n - B} |x-y|^{\alpha-n} d\mu(y) \\ &= u_1(x) + u_2(x). \end{aligned}$$

Since u_2 is continuous on B, it suffices to show that u_1 is α-quasicontinuous on B. Thus we may assume that μ has compact support. Let $\varepsilon > 0$. If we set

$$G_1 = \{x : U_\alpha \mu(x) > 2\mu(\mathbf{R}^n)/\varepsilon\},$$

then G_1 is open and

$$C_\alpha(G_1) \leqq \varepsilon/2.$$

Take a compact set $K_1 \subseteqq G_1$ for which $\mu(G_1 - K_1) < \varepsilon 4^{-2}$. Since $U_\alpha \mu$ is finite on $\mathbf{R}^n - G_1$, in view of Theorem 1.5, we can find a compact set $F_1 \subseteqq \mathbf{R}^n - G_1$ such that $\mu(\mathbf{R}^n - G_1 - F_1) < \varepsilon 4^{-2}$ and $U_\alpha(\mu|_{F_1})$ is continuous on $\mathbf{R}^n$. Define

$$\nu_1 = \mu|_{K_1 \cup F_1}$$

and

$$\mu_1 = \mu - \nu_1.$$

Then $U_\alpha \nu_1$ is continuous on $\mathbf{R}^n - G_1$. Moreover,

$$\mu_1(\mathbf{R}^n) \leqq 2\varepsilon 4^{-2}$$

and, for $x \in \mathbf{R}^n - G_1$, we have

$$U_\alpha \nu_1(x) \leqq U_\alpha \mu(x) \leqq 2\mu(\mathbf{R}^n)/\varepsilon.$$

Next consider

$$G_2 = \{x : U_\alpha \mu_1(x) > 2^2 \mu_1(\mathbf{R}^n)/\varepsilon\}.$$

Then G_2 is open and

$$C_\alpha(G_2) \leqq \varepsilon 2^{-2}.$$

As above, take a compact set $K_2 \subseteqq G_2$ for which $\mu_1(G_2 - K_2) < \varepsilon 4^{-3}$. Further, take a compact set $F_2 \subseteqq \mathbf{R}^n - G_2$ such that $\mu_1(\mathbf{R}^n - G_2 - F_2) < \varepsilon 4^{-3}$ and $U_\alpha \mu_1|_{F_2}$ is continuous on $\mathbf{R}^n$. Define

$$\nu_2 = \mu_1|_{K_2 \cup F_2}$$

and

$$\mu_2 = \mu_1 - \nu_2.$$

Then $U_\alpha \nu_2$ is continuous on $\mathbf{R}^n - G_2$. Moreover,

$$\mu_2(\mathbf{R}^n) \leqq 2\varepsilon 4^{-3}$$

and, for $x \in \mathbf{R}^n - G_2$, we have

$$U_\alpha \nu_2(x) \leqq U_\alpha \mu_1(x) \leqq 2^2 \mu_1(\mathbf{R}^n)/\varepsilon < 2^{-1}.$$

By induction, we can find $\{G_j\}$, $\{K_j\}$, $\{F_j\}$, $\{\nu_j\}$ and $\{\mu_j\}$ such that

$$G_{j+1} = \{x : U_\alpha \mu_j(x) > 2^{j+1} \mu_j(\mathbf{R}^n)/\varepsilon\},$$

$$K_{j+1} \subseteqq G_{j+1}, \qquad F_{j+1} \subseteqq \mathbf{R}^n - G_{j+1},$$
$$\mu_j(G_{j+1} - K_{j+1}) < \varepsilon 4^{-j-2},$$
$$\mu_j(\mathbf{R}^n - G_{j+1} - F_{j+1}) < \varepsilon 4^{-j-2},$$

$$\nu_{j+1} = \mu_j|_{K_{j+1}\cup F_{j+1}},$$
$$\mu_{j+1} = \mu_j - \nu_{j+1},$$
$$\mu_j(\mathbf{R}^n) < 2\varepsilon 4^{-j-1}$$

and, moreover, $U_\alpha \nu_{j+1}$ is continuous on $\mathbf{R}^n - G_{j+1}$. Then

$$C_\alpha(G_{j+1}) \leqq \varepsilon 2^{-j-1}$$

and, for $x \in \mathbf{R}^n - G_{j+1}$, we have

$$U_\alpha \nu_{j+1}(x) \leqq U_\alpha \mu_j(x) \leqq 2^{j+1}\mu_j(\mathbf{R}^n)/\varepsilon < 2^{j+1}\cdot 2\cdot 4^{-j-1} = 2^{-j}.$$

Since $\{\mu_j(\mathbf{R}^n)\}$ tends to zero as $j \to \infty$, $\mu = \nu_1 + \nu_2 + \cdots$ and

$$U_\alpha\mu = U_\alpha\nu_1 + U_\alpha\nu_2 + \cdots < U_\alpha\nu_1 + \sum_{j=2}^{\infty} 2^{-j}$$

on $\mathbf{R}^n - G$ with $G = \bigcup_{j=1}^{\infty} G_j$. Thus we see that $U_\alpha\nu_1 + U_\alpha\nu_2 + \cdots$ converges to $U_\alpha\mu$ uniformly on $\mathbf{R}^n - G$. Since each $U_\alpha\nu_j$ is continuous, $U_\alpha\mu$ is continuous as a function on $\mathbf{R}^n - G$. Finally we have only to note that

$$C_\alpha(G) \leqq \sum_{j=1}^{\infty} C_\alpha(G_j) \leqq \sum_{j=1}^{\infty} \varepsilon 2^{-j} = \varepsilon.$$

THEOREM 4.5. *Let $\{\mu_j\}$ be a sequence of measures on $\mathbf{R}^n$ which converges vaguely to μ_0. If $\{\mu_j(\mathbf{R}^n)\}$ is bounded, then*

$$\liminf_{j\to\infty} U_\alpha\mu_j(x) = U_\alpha\mu_0(x)$$

holds for every x except those in a set of α-capacity zero.

PROOF. For positive integers j and k, consider the sets

$$A_{j,k} = \{x : U_\alpha\mu_0(x) + 1/k \leqq U_\alpha\mu_j(x)\}$$

and

$$E_{\ell,k} = \bigcap_{j=\ell}^{\infty} A_{j,k}.$$

Since α-potentials are all α-quasicontinuous by Theorem 4.4, there exists an open set ω_j such that $C_\alpha(\omega_j) < 2^{-j}$ and $U_\alpha\mu_0$ and $U_\alpha\mu_j$ are continuous as a function on $\mathbf{R}^n - \omega_j$. Hence $A_{j,k} - \omega_j$ is closed, so that $E_{\ell,k} - G_\ell$ is closed, where $G_\ell = \bigcup_{j=\ell}^{\infty} \omega_j$. Note here that

$$C_\alpha(G_\ell) < \sum_{j=\ell}^{\infty} 2^{-j} = 2^{-\ell+1}.$$

Let ν be a measure with compact support in $E_{\ell,k}$ such that $U_\alpha \nu$ is continuous. Then, since $\{\mu_j(\mathbf{R}^n)\}$ is bounded, we see that

$$\begin{aligned}\int d\nu(x) &\leqq \int k[U_\alpha \mu_j(x) - U_\alpha \mu_0(x)] d\nu(x) \\ &= k\left(\int U_\alpha \nu(y)\, d\mu_j(y) - \int U_\alpha \nu(y)\, d\mu_0(y)\right) \\ &\to 0 \qquad \text{as } j \to \infty ,\end{aligned}$$

which implies that $\nu(\mathbf{R}^n) = 0$, so that $c_\alpha(E_{\ell,k}) = 0$. Since $E_{\ell,k} - G_\ell$ is closed, Theorem 4.3, together with the countable subadditivity of C_α, gives

$$C_\alpha(E_{\ell,k} - G_\ell) = 0.$$

Thus it follows that

$$C_\alpha(E_{\ell,k}) \leqq C_\alpha(E_{\ell,k} - G_\ell) + C_\alpha(G_\ell) < 2^{-\ell+1}.$$

Noting that $C_\alpha(E_{\ell,k}) \leqq C_\alpha(E_{m,k}) < 2^{-m+1}$ for any $m > \ell$, we have

$$C_\alpha(E_{\ell,k}) = 0.$$

Now consider the sets

$$E_k = \bigcup_{\ell=1}^{\infty} E_{\ell,k} \qquad \text{and} \qquad E = \bigcup_{k=1}^{\infty} E_k.$$

Then we see that $C_\alpha(E) = 0$ and

$$E = \left\{x : \liminf_{j\to\infty} U_\alpha \mu_j(x) > U_\alpha \mu_0(x)\right\}.$$

Thus the theorem is proved.

THEOREM 4.6. *If $\{E_j\}$ is a nondecreasing sequence of sets in $\mathbf{R}^n$, then*

$$\lim_{j\to\infty} C_\alpha(E_j) = C_\alpha(E), \qquad E = \bigcup_{j=1}^{\infty} E_j.$$

PROOF. We have only to show the case where the left hand-side is finite. For each positive integer j, take a measure μ_j such that

$$U_\alpha \mu_j(x) \geqq 1 \qquad \text{for all } x \in E_j$$

and

$$\mu_j(\mathbf{R}^n) < C_\alpha(E_j) + 1/j.$$

Since $\{C_\alpha(E_j)\}$ is bounded by assumption, $\{\mu_j(\mathbf{R}^n)\}$ is also bounded and we may assume that $\{\mu_j\}$ converges vaguely to a measure μ_0. Then, in view of Theorem 4.5,

$$U_\alpha\mu_0(x) \geqq 1$$

for all $x \in E$ except those in a set of C_α-capacity zero. Thus we have

$$C_\alpha(E) \leqq \mu_0(\mathbf{R}^n) \leqq \liminf_{j\to\infty} \ \mu_j(\mathbf{R}^n) \leqq \liminf_{j\to\infty} \ C_\alpha(E_j) \leqq C_\alpha(E),$$

which proves the required conclusion.

Finally we give the capacitability result.

THEOREM 4.7. *If S is a Suslin set in $\mathbf{R}^n$, then*

$$C_\alpha(S) = c_\alpha(S). \tag{4.3}$$

PROOF. By Theorem 4.3, every compact set is capacitable, that is, (4.3) holds for every compact sets. On the other hand, Theorem 4.6 implies that C_α has the increasing property. Now Theorem 3.2 and its proof in Chapter 1 give the capacitability result.

2.5 Fine limits

For a set E, a point x_0 and $r > 0$, set

$$E(x_0, r) = E \cap B(x_0, r).$$

We say that E is α-thin at x_0 if

$$\sum_{j=1}^{\infty} 2^{j(n-\alpha)} C_\alpha(E(x_0, 2^{-j})) < \infty.$$

LEMMA 5.1. *A set E is α-thin at x_0 if and only if*

$$\int_0^1 [r^{\alpha-n} C_\alpha(E(x_0, r))] \ dr/r < \infty.$$

REMARK 5.1. We note that E is α-thin at x_0 if and only if

$$\sum_{j=1}^{\infty} 2^{j(n-\alpha)} C_\alpha(\hat{E}(x_0, 2^{-j+1})) < \infty,$$

where $\hat{E}(x_0, 2^{-j+1}) = E \cap B(x_0, 2^{-j+1}) - B(x_0, 2^{-j})$.

We say that a function f has an α-fine limit a at x_0 if

$$\lim_{x \to x_0, x \notin E} f(x) = a$$

for a set E which is α-thin at x_0; f is said to be α-finely continuous at x_0 if it has an α-fine limit $f(x_0)$ at x_0.

THEOREM 5.1. *For any $\mu \in \mathcal{M}$, $U_\alpha\mu$ has an α-fine limit $U_\alpha\mu(x_0)$ at any point x_0, that is, $U_\alpha\mu$ is α-finely continuous everywhere in the extended sense.*

PROOF. Without loss of generality, we may assume that x_0 is the origin. Further, we may assume that $U_\alpha\mu(0) < \infty$, since if otherwise, then the lower semicontinuity implies that

$$\lim_{x \to 0} U_\alpha\mu(x) = \infty = U_\alpha\mu(0).$$

For $x \neq 0$, set

$$\begin{aligned} u_1(x) &= \int_{B(x,|x|/2)} |x-y|^{\alpha-n}\, d\mu(y), \\ u_2(x) &= \int_{\mathbf{R}^n - B(x,|x|/2)} |x-y|^{\alpha-n}\, d\mu(y). \end{aligned}$$

If $|x-y| \geqq |x|/2$, then $|y| \leqq |x-y| + |x| \leqq 3|x-y|$, so that Lebesgue's dominated convergence theorem gives

$$\lim_{x \to 0} u_2(x) = U_\alpha\mu(0). \tag{5.1}$$

Since $U_\alpha\mu(0) < \infty$, we can find a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j 2^{j(n-\alpha)} \mu(G_j) < \infty, \tag{5.2}$$

where $G_j = \{x : 2^{-j-1} < |x| < 2^{-j+2}\}$. Letting $B_j = B(0, 2^{-j+1}) - B(0, 2^{-j})$, we define

$$E_j = \{x \in B_j : u_1(x) > a_j^{-1}\}.$$

If $x \in E_j$, then $B(x, |x|/2) \subseteqq G_j$ and thus

$$a_j^{-1} < u_1(x) \leqq \int_{G_j} |x-y|^{\alpha-n}\, d\mu(y),$$

so that it follows from the definition of C_α that

$$C_\alpha(E_j) \leqq a_j \mu(G_j).$$

Consider the set $E = \bigcup_{j=1}^{\infty} E_j$. Then $\hat{E}(0, 2^{-j+1}) = E_j$, and we see from (5.2) that the set $E = \bigcup_{j=1}^{\infty} E_j$ is α-thin at 0. Further, if $x \in B_j - E$, then $u_1(x) \leqq a_j^{-1}$, which proves

$$\limsup_{x \to 0, x \notin E} u_1(x) \leqq \lim_{j \to \infty} a_j^{-1} = 0.$$

This implies that

$$\lim_{x \to 0, x \notin E} u_1(x) = 0.$$

In view of (5.1), it follows that

$$\lim_{x \to 0, x \notin E} U_\alpha \mu(x) = U_\alpha \mu(0),$$

as required.

THEOREM 5.2. *If E is α-thin at 0, then there exists a measure μ such that $U_\alpha \mu(0) < \infty$ and*

$$\lim_{x \to 0, x \in E} U_\alpha \mu(x) = \infty. \tag{5.3}$$

PROOF. Since E is α-thin at 0, we can find a sequence $\{a_j\}$ of positive numbers such that $\lim_{j \to \infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j 2^{j(n-\alpha)} C_\alpha(E_j) < \infty, \tag{5.4}$$

where $E_j = \hat{E}(0, 2^{-j+1})$. For each positive integer j, by the definition of C_α, there exists a measure μ_j such that $\mu_j(\mathbf{R}^n) < C_\alpha(E_j) + a_j^{-1} 2^{-j} 2^{-j(n-\alpha)}$ and

$$U_\alpha \mu_j(x) \geqq 1 \qquad \text{whenever } x \in E_j\ .$$

If $x \in B_j = B(0, 2^{-j+1}) - B(0, 2^{-j})$, then

$$\begin{aligned} \int_{\mathbf{R}^n - B(x, |x|/2)} |x-y|^{\alpha-n}\, d\mu_j(y) &\leqq (|x|/2)^{\alpha-n}\, \mu_j(\mathbf{R}^n) \\ &\leqq 2^{n-\alpha} \{ 2^{j(n-\alpha)} C_\alpha(E_j) + a_j^{-1} 2^{-j} \}. \end{aligned}$$

Hence, if j is large enough, then

$$\int_{\mathbf{R}^n - B(x, |x|/2)} |x-y|^{\alpha-n}\, d\mu_j(y) < 1/2$$

and it follows from (5.4) that

$$\int_{G_j} |x-y|^{\alpha-n}\, d\mu_j(x) \geqq \int_{B(x,|x|/2)} |x-y|^{\alpha-n}\, d\mu_j(y) \geqq 1/2$$

whenever $x \in E_j$, where $G_j = \{x : 2^{-j-1} < |x| < 2^{-j+2}\}$ as before. Now consider

$$\mu = \sum_{j=1}^{\infty} a_j \mu_j|_{G_j}.$$

Then (5.4) gives

$$\begin{aligned} U_\alpha\mu(0) &\leqq \sum_{j=1}^{\infty} a_j \int_{G_j} |y|^{\alpha-n}\, d\mu_j \\ &\leqq 2^{n-\alpha} \sum_{j=1}^{\infty} a_j 2^{j(n-\alpha)} \mu_j(G_j) \\ &\leqq 2^{n-\alpha} \sum_{j=1}^{\infty} \{a_j 2^{j(n-\alpha)} C_\alpha(E_j) + 2^{-j}\} < \infty. \end{aligned}$$

On the other hand, if $x \in E_j$, then

$$\int |x-y|^{\alpha-n}\, d\mu(y) \geqq a_j \int_{G_j} |x-y|^{\alpha-n}\, d\mu_j(y) \geqq a_j/2,$$

so that (5.3) follows.

THEOREM 5.3. *For any $\mu \in \mathcal{M}$, if $U_\alpha\mu \not\equiv \infty$, then $|x-x_0|^{n-\alpha} U_\alpha\mu(x)$ has α-fine limit $\mu(\{x_0\})$ at any point x_0.*

PROOF. As in the proof of Theorem 5.1, we may assume that x_0 is the origin, and write

$$U_\alpha\mu(x) = u_1(x) + u_2(x).$$

Since $|x|/|x-y| \leqq 2$ if $y \in \mathbf{R}^n - B(x,|x|/2)$, we see by Lebesgue's dominated convergence theorem that

$$\lim_{x \to 0} |x|^{n-\alpha} u_2(x) = \mu(\{0\}). \tag{5.5}$$

Let $\{a_j\}$ be a sequence of positive numbers such that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j \mu(G_j) < \infty, \tag{5.6}$$

where $G_j = \{x : 2^{-j-1} < |x| < 2^{-j+2}\}$. Letting $B_j = B(0,2^{-j+1}) - B(0,2^{-j})$ as before, we define

$$E_j = \{x \in B_j : u_1(x) > a_j^{-1} 2^{j(n-\alpha)}\}.$$

If $x \in E_j$, then

$$2^{j(n-\alpha)} a_j^{-1} < u_1(x) \leqq \int_{G_j} |x-y|^{\alpha-n} \, d\mu(y),$$

so that it follows from the definition of C_α that

$$C_\alpha(E_j) \leqq a_j 2^{-j(n-\alpha)} \mu(G_j).$$

If we set $E = \bigcup_{j=1}^{\infty} E_j$, then E is seen to be α-thin at 0, on account of (5.6). Further, as in the proof of Theorem 5.1, we see that

$$\lim_{x \to 0, x \notin E} |x|^{n-\alpha} u_1(x) = 0.$$

In view of (5.5), it follows that

$$\lim_{x \to 0, x \notin E} |x|^{n-\alpha} U_\alpha \mu(x) = \mu(\{0\}),$$

as required.

The following can be proved in the same way as Theorem 5.2.

THEOREM 5.4. *If E is α-thin at 0, then there exists a measure μ such that $U_\alpha \mu \not\equiv \infty$ and*

$$\lim_{x \to 0, x \in E} |x|^{n-\alpha} U_\alpha \mu(x) = \infty.$$

2.6 Logarithmic potentials

For a measure μ on $\mathbf{R}^n$, we define the logarithmic potential of μ by setting

$$U_n \mu(x) = \int U_n(x-y) \, d\mu(y) = \int \log \frac{1}{|x-y|} \, d\mu(y).$$

THEOREM 6.1. *For any measure μ, the following assertions are equivalent:*

(6.1) $$-\infty < U_n \mu \not\equiv \infty;$$

(6.2) $$\int \log(2+|y|) \, d\mu(y) < \infty.$$

PROOF. Write

$$\begin{aligned} U_n \mu(x) &= \int_{B(x,1)} \log \frac{1}{|x-y|} \, d\mu(y) - \int_{\mathbf{R}^n - B(x,1)} \log |x-y| \, d\mu(y) \\ &= u^+(x) - u^-(x). \end{aligned}$$

First note that if $u^-(x) < \infty$ for some x, then (6.2) holds. Further, if (6.2) holds, then $u^-(x) < \infty$ for all x. On the other hand,

$$\begin{aligned}\int_{B(0,N)} u^+(x)\, dx &= \int_{B(0,N+1)} \left(\int_{B(y,1)} \log \frac{1}{|x-y|}\, dx \right) d\mu(y) \\ &\leqq M\mu(B(0,N+1)) < \infty,\end{aligned}$$

which implies that $u^+ \in L^1_{loc}(\mathbf{R}^n)$. Thus (6.1) and (6.2) are equivalent to each other.

REMARK 6.1. If (6.1) or (6.2) of Theorem 6.1 holds, then $U_n\mu$ is lower semicontinuous on $\mathbf{R}^n$.

For a set $E \subseteqq \mathbf{B}$, we define a logarithmic capacity by setting

$$C_n^{(1)}(E) = \inf\ \mu(\mathbf{B}),$$

where the infimum is taken over all measures μ such that

$$\int \log \frac{2}{|x-y|}\, d\mu(y) \geqq 1 \qquad \text{for any } x \in E.$$

Then it is easy to see that $C_n^{(1)}$ is a nondecreasing, countably subadditive, and outer capacity on the family of subsets of the unit ball $\mathbf{B}$. A set E is logarithmically thin, or simply, n-thin at $x_0 \in \mathbf{B}$ if

$$\sum_{j=1}^{\infty} jC_n^{(1)}(\hat{E}(x_0, 2^{-j+1})) < \infty,$$

where $\hat{E}(x_0, 2^{-j+1}) = E \cap B(x_0, 2^{-j+1}) - B(x_0, 2^{-j})$ as before. A function f has an n-fine limit ℓ at x_0 if $f(x) \to \ell$ as x tends to x_0, except for a set E which is n-thin at x_0.

THEOREM 6.2. *For any $\mu \in \mathcal{M}(\mathbf{B})$ satisfying* (6.2), *$U_n\mu$ has an n-fine limit $U_n\mu(x_0)$ at any point $x_0 \in \mathbf{B}$.*

PROOF. Since $U_n\mu$ is lower semicontinuous, it suffices to treat the case when $U_n\mu(x_0) < \infty$. For $x \neq x_0$, write $U_n\mu(x) = u_1(x) + u_2(x)$, where

$$\begin{aligned}u_1(x) &= \int_{B(x,|x-x_0|/2)} \log \frac{1}{|x-y|}\, d\mu(y), \\ u_2(x) &= \int_{\mathbf{B}-B(x,|x-x_0|/2)} \log \frac{1}{|x-y|}\, d\mu(y).\end{aligned}$$

If $|x-y| \geqq |x-x_0|/2$, then $|x_0-y| \leqq |x_0-x| + |x-y| \leqq 3|x-y|$, so that Lebesgue's dominated convergence theorem gives

$$\lim_{x \to x_0} u_2(x) = U_n\mu(x_0). \tag{6.3}$$

Since $U_n\mu(x_0) < \infty$,

$$\sum_{j=1}^{\infty} j\ \mu(G_j) < \infty,$$

where $G_j = \{x : 2^{-j-1} < |x - x_0| < 2^{-j+2}\}$. Hence we can find a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} ja_j\mu(G_j) < \infty. \tag{6.4}$$

Letting $B_j = B(x_0, 2^{-j+1}) - B(x_0, 2^{-j})$, we consider $E_j = \{x \in B_j : u_1(x) > a_j^{-1}\}$ and $E = \bigcup_{j=1}^{\infty} E_j$. Note that if $x \in E_j$ and $j \geqq 2$, then

$$u_1(x) \leqq \int_{G_j} \log \frac{2}{|x-y|}\ d\mu(y),$$

so that

$$C_n^{(1)}(E_j) \leqq a_j\mu_j(G_j).$$

Thus we see from (6.4) that E is n-thin at x_0 and

$$0 \leqq \limsup_{x\to x_0, x\in\mathbf{R}^n-E} u_1(x) \leqq \limsup_{j\to\infty}\ a_j^{-1} = 0.$$

In view of (6.3), it follows that

$$\lim_{x\to x_0, x\in\mathbf{R}^n-E} U_n\mu(x) = U_n\mu(x_0),$$

as required.

In the same manner as Theorem 5.3, we can prove the following result.

THEOREM 6.3. *Let $\mu \in \mathcal{M}(\mathbf{B})$ satisfy* (6.2). *Then, for $x_0 \in \mathbf{B}$, there exists a set E which is n-thin at x_0 and satisfies*

$$\lim_{x\to x_0, x\in\mathbf{R}^n-E} \left[\log \frac{1}{|x-x_0|}\right]^{-1} U_n\mu(x) = \mu(\{x_0\}).$$

REMARK 6.2. For $R > 0$ and a set $E \subseteqq B(0,R)$, we define

$$C_n^{(R)}(E) = \inf\ \mu(\mathbf{R}^n),$$

where the infimum is taken over all measures $\mu \in \mathcal{M}(B(0,R))$ such that

$$\int \log \frac{2R}{|x-y|}\ d\mu(y) \geqq 1 \qquad \text{for any } x \in E.$$

Then we have the following results.

(1) $C_n^{(R)}$ is an nondecreasing, countably subadditive, and outer capacity on the family of subsets of $B(0,R)$.

(2) For $E \subseteqq B(0,R)$ and $R' > R$, $C_n^{(R)}(E) = 0$ if and only if $C_n^{(R')}(E) = 0$.

(3) Let $E \subseteqq B(0,R)$, $x_0 \in B(0,R)$ and $0 < R < R'$. Then

$$\sum_{j=1}^{\infty} jC_n^{(R)}(E_j) < \infty \qquad \text{if and only if} \qquad \sum_{j=1}^{\infty} jC_n^{(R')}(E_j) < \infty,$$

where $E_j = \hat{E}(x_0, 2^{-j+1}) = \{x \in E : 2^{-j} \leqq |x - x_0| < 2^{-j+1}\}$; in this case, E will be called n-thin at x_0 (with respect to $C_n^{(R)}$).

(4) Let $x_0 \in \mathbf{B}$. Then a set E is n-thin at x_0 if and only if

$$\sum_{j=1}^{\infty} jC_n^{(2^{-j+2})}(\hat{E}(x_0, 2^{-j+1})) < \infty.$$

We show only (4). For this purpose, take $\mu \in \mathcal{M}(\mathbf{B})$ such that

$$\int \log \frac{2}{|x-y|}\, d\mu(y) \geqq 1 \qquad \text{for any } x \in E_j = \hat{E}(x_0, 2^{-j+1}).$$

Then note that

$$\int \log \frac{2}{|x-y|}\, d\mu(y) \leqq \int_{G_j} \log \frac{2^{-j+3}}{|x-y|}\, d\mu(y) + (j+2)(\log 2)\mu(\mathbf{B}),$$

so that

$$C_n^{(2^{-j+2})}(E_j) \leqq \frac{\mu(G_j)}{1-(j+2)(\log 2)\mu(\mathbf{B})},$$

whenever $1-(j+2)(\log 2)\mu(\mathbf{B}) > 0$. This implies that

$$C_n^{(2^{-j+2})}(E_j) \leqq \frac{C_n^{(1)}(E_j)}{1-(j+2)(\log 2)C_n^{(1)}(E_j)},$$

whenever $1-(j+2)(\log 2)C_n^{(1)}(E_j) > 0$. Hence, if $jC_n^{(1)}(E_j)$ is small enough, then we have

$$C_n^{(2^{-j+2})}(E_j) \leqq 2^{-1}C_n^{(1)}(E_j).$$

Thus it follows that the n-thinness of E at x_0 implies the required inequality. The converse is trivial.

Remark 6.3. For a general set $E \subseteqq \mathbf{R}^n$, we write $C_n(E) = 0$ when

$$C_n^{(R)}(E \cap B(0,R)) = 0$$

for any $R > 0$; in this case, E is said to be of logarithmic capacity zero. Note that $C_n(E) = 0$ if and only if there exists a measure μ_0 on $\mathbf{R}^n$ satisfying (6.2) such that $U_n\mu_0(x) = \infty$ for any $x \in E$.

THEOREM 6.4. *Let* $\lambda = \mu - \nu$ *for* $\mu, \nu \in \mathcal{M}(\mathbf{B})$. *If* $\lambda(\mathbf{B}) = 0$, *that is,* $\mu(\mathbf{B}) = \nu(\mathbf{B})$ *and*

$$\int\int |\log|x-y||\ d\mu(x)d\mu(y) + \int\int |\log|x-y||\ d\nu(x)d\nu(y) < \infty,$$

then

$$\int\int \log\frac{1}{|x-y|}\ d\lambda(x)d\lambda(y) = \omega_n \int \left(U_{n/2}\lambda(z)\right)^2\ dz \geqq 0.$$

PROOF. Let $R > 1$. We show that

$$\int_{B(0,R)} |x-z|^{-n/2}|z-y|^{-n/2}\ dz = \omega_n \log\frac{1}{|x-y|}\ +\ M_R\ +\ I, \tag{6.5}$$

for all $x, y \in \mathbf{B}$, where M_R is a constant and $|I(x,y)| \leqq M/R$ with a positive constant M. In fact, letting $\ell = |x-y|$ and $e = (x-y)/\ell$, we have

$$\begin{aligned}
J(R) &\equiv \int_{B(0,R)} |x-z|^{-n/2}|z-y|^{-n/2}\ dz \\
&= \int_{B(x,R)} |x-z|^{-n/2}|z-y|^{-n/2}\ dz + I_1 \\
&= \int_{B(0,R)} |z|^{-n/2}|(x-y)-z|^{-n/2}\ dz + I_1 \\
&= \int_{B(0,R/\ell)} |z|^{-n/2}|e-z|^{-n/2}\ dz + I_1 \\
&= \int_{B(0,R)} |z|^{-n/2}|e-z|^{-n/2}\ dz \\
&\quad + \int_{B(0,R/\ell)-B(0,R)} |z|^{-n/2}|e-z|^{-n/2}\ dz + I_1 \\
&= M_R + \int_{B(0,R/\ell)-B(0,R)} |z|^{-n/2}|e-z|^{-n/2}\ dz + I_1,
\end{aligned}$$

where $|I_1| \leqq M/R$ and $M_R = \int_{B(0,R)} |z|^{-n/2}|e-z|^{-n/2}\ dz$ is constant for $e \in \mathbf{S}$. On the other hand,

$$\begin{aligned}
J(R) &= M_R + \int_{B(0,R/\ell)-B(0,R)} |z|^{-n}\ dz + I_2 + I_1 \\
&= M_R + \omega_n(\log 1/\ell) + I_1 + I_2;
\end{aligned}$$

here note that

$$\begin{aligned}
|I_2| &= \left|\int_{B(0,R/\ell)-B(0,R)} \{|z|^{-n/2}|e-z|^{-n/2} - |z|^{-n}\}\ dz\right| \\
&\leqq M\int_{B(0,R/\ell)-B(0,R)} |z|^{-n-1}\ dz \leqq M/R.
\end{aligned}$$

Thus (6.5) follows. Since λ has vanishing total mass, we see that

$$\begin{aligned}&\int\int\left(\int_{B(0,R)}|x-z|^{-n/2}|z-y|^{-n/2}\,dz\right)d\lambda(x)d\lambda(y)\\ =\;&\omega_n\int\int\log\frac{1}{|x-y|}\,d\lambda(x)d\lambda(y)+\int\int I\,d\lambda(x)d\lambda(y).\end{aligned}$$

Since

$$\left|\int\int I\,d\lambda d\lambda\right|\leqq\frac{M}{R}\int\int d(\mu+\nu)d(\mu+\nu)=\frac{M}{R}\{\mu(\mathbf{B})+\nu(\mathbf{B})\}^2\to 0$$

as $R\to\infty$, the required equality now follows by Fubini's theorem.

REMARK 6.4. If $\alpha>0$, $\beta>0$ and $\alpha+\beta=n$, then

$$M^{-1}\log\frac{3R}{|x-y|}\leqq\int_{B(0,R)}|x-z|^{\alpha-n}|z-y|^{\beta-n}\,dz\leqq M\log\frac{3R}{|x-y|}$$

whenever x, $y\in B(0,R)$, with a positive constant M independent of R.

In case $n=2$, we see that

$$\int_{\mathbf{S}}\log\frac{1}{|x-y|}\,dS(y)=0\qquad\text{whenever } x\in\overline{\mathbf{B}}$$

(see Remark 2.2 in Chapter 3). Hence if μ is a positive measure with support in $\mathbf{B}$, then

$$\int\int\log\frac{1}{|x-y|}\,d\mu(x)d\mu(y)=2\pi\int(U_1\lambda(z))^2\,dz\geqq 0$$

for $d\lambda=d\mu-[\mu(\mathbf{B})/2\pi]dS|_{\mathbf{S}}$.

COROLLARY 6.1. *Let $n=2$. If μ is a nonnegative measure on $\mathbf{B}$ with energy zero, then $\mu=0$.*

2.7 Relationship between Hausdorff measures and capacities

Before giving a relationship between capacities and Hausdorff measures, we prepare several elementary results for Riesz capacities.

For a set E and $r>0$, write

$$rE=\{rx:x\in E\}.$$

THEOREM 7.1. *If $\alpha < n$, then*

$$C_\alpha(rE) = r^{n-\alpha} C_\alpha(E). \tag{7.1}$$

For logarithmic capacities,

$$C_n^{(1)}(rE) \leqq \frac{C_n^{(1)}(E)}{1 + (\log 1/r) C_n^{(1)}(E)} \tag{7.2}$$

whenever $E \subseteqq \mathbf{B}$, $rE \subseteqq \mathbf{B}$ and $1 + (\log 1/r) C_n^{(1)}(E) > 0$.

PROOF. First consider the case $\alpha < n$. For a nonnegative measure μ, we consider the measure defined by

$$\mu_r(B) = \mu(r^{-1}B) \qquad \text{for Borel sets } B.$$

Note that $U_\alpha \mu_r(rx) = r^{\alpha-n} U_\alpha \mu(x)$. Thus it follows that

$$C_\alpha(rE) \leqq r^{n-\alpha} \mu_r(\mathbf{R}^n) = r^{n-\alpha} \mu(\mathbf{R}^n),$$

which proves that

$$C_\alpha(rE) \leqq r^{n-\alpha} C_\alpha(E).$$

Apply this inequality and obtain

$$C_\alpha(E) = C_\alpha(r^{-1}(rE)) \leqq r^{-n+\alpha} C_\alpha(rE) \leqq C_\alpha(E).$$

Next consider the case $\alpha = n$. Let μ be a measure on $\mathbf{B}$ such that

$$\int \log \frac{2}{|x-y|}\, d\mu(y) \geqq 1 \qquad \text{whenever } x \in E.$$

Then, as above, we have

$$\int \log \frac{2}{|rx - z|}\, d\mu_r(z) \geqq 1 + \log \frac{1}{r}\, \mu(\mathbf{B})$$

for $x \in E$. Hence it follows that

$$C_n(rE) \leqq \frac{\mu(\mathbf{B})}{1 + (\log 1/r) \mu(\mathbf{B})},$$

which implies (7.2).

It is convenient to define

$$U_\alpha(r) = \begin{cases} r^{\alpha-n}, & \alpha < n, \\ \log \dfrac{1}{r}, & \alpha = n, \end{cases}$$

and

$$U_\alpha(x) = U_\alpha(|x|).$$

THEOREM 7.2. $0 < C_\alpha(\mathbf{B}) < \infty$.

In fact, if $\alpha < n$, then we note that

$$\int_{\mathbf{B}} |x-y|^{\alpha-n} dy \geqq 2^{\alpha-n}|\mathbf{B}| = c \qquad \text{for any } x \in \mathbf{B},$$

so that

$$C_\alpha(\mathbf{B}) \leqq c^{-1}|\mathbf{B}| = 2^{n-\alpha}.$$

On the other hand, in view of Remark 1.2, we see that

$$\begin{aligned} \int_{\mathbf{B}} |x-y|^{\alpha-n} dy &\leqq \int_{B(x,1)} |x-y|^{\alpha-n} dy \\ &= \int_{\mathbf{B}} |y|^{\alpha-n} dy = \omega_n/\alpha \equiv d \end{aligned}$$

for any x. Hence,

$$c_\alpha(\mathbf{B}) \geqq d^{-1}|\mathbf{B}| > 0.$$

The logarithmic case $\alpha = n$ can be proved similarly.

COROLLARY 7.1. *In case $\alpha < n$,*

$$C_\alpha(B(x,r)) = c_\alpha r^{n-\alpha}$$

for any ball $B(x,r)$ with $c_\alpha = C_\alpha(\mathbf{B})$. In the logarithmic case, there exists $c_n > 0$ such that

$$c_n^{-1}[U_n(r)]^{-1} \leqq C_n^{(1)}(B(x,r)) \leqq c_n[U_n(r)]^{-1}$$

for any ball $B(x,r)$ such that $B(x,r) \subseteqq B(0,1/2)$.

THEOREM 7.3 (Dini's theorem). *Let f_j, $j = 1,2,...$, be upper semicontinuous on a compact set K. If $f_j(x)$ decreases to 0 for each $x \in K$, then $f_j(x)$ converges to 0 uniformly on K.*

PROOF. For $\varepsilon > 0$, set $G_j = \{x \in K : f_j(x) < \varepsilon\}$. Then G_j is relatively open in K, $G_j \subseteqq G_{j+1}$ and $\bigcup_{j=1}^{\infty} G_j \supseteqq K$. Hence by the compactness of K there exists j_0 for which $G_{j_0} \supseteqq K$, which means that

$$f_j(x) < \varepsilon \qquad \text{for all } x \in K \text{ and } j \geqq j_0 \,.$$

Thus the convergence is uniform on K.

THEOREM 7.4. *If $U_\alpha\mu$ is continuous on a compact set K, then there exists a function k such that*

$$\lim_{r\to 0} k(r)/U_\alpha(r) = \infty \tag{7.3}$$

and the potential $\int k(|x-y|)d\mu(y)$ is bounded on K.

PROOF. Consider the functions

$$u_j(x) = \int \min\{U_\alpha(x-y), j\}d\mu(y),$$

which are continuous on K. Since u_j increases to a continuous function $U_\alpha\mu$ pointwise on K, Dini's theorem implies that the convergence is uniform on K. Hence, for any positive integer m, find $j(m)$ such that

$$\int k_m(|x-y|)d\mu(y) < 2^{-m} \qquad \text{for all } x \in K,$$

where $k_m(r) = U_\alpha(r) - \min\{U_\alpha(r), j(m)\}$. Now consider the function

$$k(r) = \sum_{m=1}^{\infty} mk_m(r).$$

Then

$$\liminf_{r\to 0} k(r)/U_\alpha(r) \geqq \liminf_{r\to 0} mk_m(r)/U_\alpha(r) = m$$

for any m, which implies (7.3). On the other hand,

$$\int k(|x-y|)d\mu(y) = \sum_{m=1}^{\infty} m\int k_m(|x-y|)d\mu(y) < \sum_{m=1}^{\infty} m2^{-m} < \infty$$

for any $x \in K$.

THEOREM 7.5. *If $H_{[U_\alpha]^{-1}}(E) < \infty$, then $C_\alpha(E) = 0$.*

PROOF. We prove this theorem in case $\alpha < n$. For simplicity, write $h_\alpha = H_{[U_\alpha]^{-1}}$. If $h_\alpha(E) = 0$, then it is not difficult to show that $C_\alpha(E) = 0$. In fact, for given $\varepsilon > 0$, there exists a covering $\{B(x_j, r_j)\}$ of E such that $r_j < \varepsilon$ and

$$\sum_{j=1}^{\infty} [U_\alpha(r_j)]^{-1} < \varepsilon.$$

By a covering lemma (see Theorem 10.1 in Chapter 1), choose a disjoint subfamily $\{B(x_{j(m)}, r_{j(m)})\}$ such that $\{B(x_{j(m)}, 5r_{j(m)})\}$ covers E. Hence

$$\begin{aligned} C_\alpha(E) &\leqq C_\alpha(\bigcup_{m=1}^{\infty} B(x_{j(m)}, 5r_{j(m)})) \\ &\leqq \sum_{m=1}^{\infty} C_\alpha(B(x_{j(m)}, 5r_{j(m)})) \\ &\leqq c_\alpha \sum_{m=1}^{\infty} [U_\alpha(5r_{j(m)})]^{-1} \leqq M\varepsilon, \end{aligned}$$

which shows that $C_\alpha(E) = 0$. Now we show the general case. First find a G_δ set E^* such that $E \subseteqq E^*$ and $h_\alpha(E^*) < \infty$. We show that $C_\alpha(K) = 0$ when assuming that $0 < h_\alpha(K) < \infty$, for a compact set $K \subseteqq E^*$. If $C_\alpha(K) > 0$, then Theorems 1.5 and 4.3 give a continuous potential $U_\alpha\mu$ of a positive measure μ supported by K. Now take a kernel function k as in Theorem 7.4, and consider the capacities

$$C_k(A) = \inf \left\{\mu(\mathbf{R}^n) : \int k(|x-y|)\, d\mu(y) \geqq 1 \quad \text{for all } x \in A\right\}$$

and

$$c_k(A) = \sup \left\{\nu(\mathbf{R}^n) : S_\nu \subseteqq A \ \text{ and } \int k(|x-y|)\, d\mu(y) \leqq 1 \ \text{ for all } x \in \mathbf{R}^n\right\},$$

in the same manner as α-capacities. We see here that

$$c_k(K) > 0$$

and

$$C_k(e) = c_k(e) \qquad \text{whenever } e \text{ is compact}$$

as in Theorem 4.3, through minimax lemma. We show below that $C_k(K) = 0$ on the contrary. For this purpose, if $\delta > 0$ is given, then take a covering $\{B(x_j, r_j)\}$ of K such that $0 < r_j < \delta$ and

$$\sum_j [U_\alpha(r_j)]^{-1} < 2h_\alpha(K).$$

Letting $M(r) = \inf_{0<t<r} k(t)/U_\alpha(t)$, we note that

$$\int_{B(x,r)} k(|z-y|)\, dy \geqq M(2r)U_\alpha(2r)|B(x,r)| \qquad \text{whenever } z \in B(x,r),$$

so that

$$C_k(B(x,r)) \leqq [M(2r)U_\alpha(2r)|B(x,r)|]^{-1} \int_{B(x,r)} dy = [M(2r)U_\alpha(2r)]^{-1}.$$

It follows that

$$C_k(K) \leqq \sum_j C_k(B(x_j,r_j)) \leqq [M(2\delta)]^{-1} \sum_j [U_\alpha(2r_j)]^{-1} \leqq M[M(2\delta)]^{-1} h_\alpha(K),$$

which proves by letting $\delta \to 0$

$$C_k(K) = 0.$$

Thus a contradiction follows, and hence $C_\alpha(K) = 0$ as required.

We recall that a function h is a measure function if h is positive nondecreasing on the interval $(0, \infty)$ and satisfies the doubling condition.

THEOREM 7.6. *Let h be a measure function such that*

$$\int_0^1 h(r)r^{\alpha-n-1}dr < \infty. \tag{7.4}$$

If $C_\alpha(E) = 0$, then $H_h(E) = 0$.

PROOF. Suppose $\alpha < n$ and E is a Borel set with $H_h(E) > 0$. Then there exists a compact subset K of E such that $H_h(K) > 0$. In view of Frostman's theorem, we can find a positive measure μ such that

$$\mu(B(x,r)) \leqq h(r) \qquad \text{for any ball } B(x,r).$$

Then we see that

$$\begin{aligned} \int U_\alpha(x-y)\, d\mu(y) &= \int_0^\infty U_\alpha(r)\, d\mu(B(x,r)) \\ &= \int_0^\infty \mu(B(x,r))\, d(-U_\alpha(r)), \end{aligned}$$

which is bounded by (7.4). Thus $c_\alpha(K) > 0$, and hence $C_\alpha(K) > 0$ by Theorem 4.3. Now a contradiction follows.

COROLLARY 7.2. *If $C_\alpha(E) = 0$, then $H_{n-\alpha+\delta}(E) = 0$ for any $\delta > 0$; in this case, E is said to have Hausdorff dimension at most $n - \alpha$.*

2.8 Radial limits

As applications of fine limit result, we give radial limit results. For this purpose, we need the following lemma.

LEMMA 8.1. *Let f be a Lipschitz mapping from a compact set K to a compact set $K' = f(K)$, that is,*

$$|f(x) - f(y)| \leqq A|x-y| \qquad \text{whenever } x, y \in K,$$

where A is a positive constant. If $E \subseteqq K$ and $\alpha < n$, then

$$C_\alpha(f(E)) \leqq A^{n-\alpha}C_\alpha(E); \tag{8.1}$$

in the logarithmic case,

$$C_n^{(2)}(f(E)) \leqq MC_n^{(2)}(E)$$

with a positive constant M, whenever $E \subseteqq K \subseteqq B(0,2)$ and $K' \subseteqq B(0,2)$.

PROOF. We show the case $\alpha < n$ only. Let μ be a measure such that

$$U_\alpha\mu(x) \geqq 1 \qquad \text{whenever } x \in E.$$

If we define a measure ν be setting

$$\nu(X) = \mu(f^{-1}(X)) \qquad \text{for } X \subseteqq \mathbf{R}^n,$$

then we see that

$$U_\alpha \nu(f(x)) \geqq A^{\alpha-n} U_\alpha \mu(x),$$

so that

$$C_\alpha(f(E)) \leqq A^{n-\alpha} \nu(\mathbf{R}^n) = A^{n-\alpha} \mu(\mathbf{R}^n).$$

Thus (8.1) follows.

For a set E, we define the radial projection to $\mathbf{S}$ by

$$\tilde{E} = \{\xi \in \mathbf{S} : r\xi \in E \text{ for some } r > 0\}.$$

LEMMA 8.2. *If E is α-thin at the origin, then*

$$C_\alpha\Big(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \tilde{E}_j\Big) = 0, \tag{8.2}$$

where $E_j = E \cap B(0, 2^{-j+1}) - B(0, 2^{-j})$.

PROOF. We show the case $\alpha < n$ only. In this case, by Theorem 7.1,

$$C_\alpha(2^j E_j) = 2^{j(n-\alpha)} C_\alpha(E_j).$$

By considering the contraction mapping: $x \to x/|x|$ from $\mathbf{R}^n - \mathbf{B}$ to $\mathbf{S}$, Lemma 8.1 yields

$$C_\alpha(\tilde{E}_j) = C_\alpha((2^j E_j)^{\sim}) \leqq C_\alpha(2^j E_j).$$

Hence it follows that

$$C_\alpha(\tilde{E}_j) \leqq 2^{j(n-\alpha)} C_\alpha(E_j)$$

and then by the subadditivity of C_α

$$C_\alpha\Big(\bigcup_{j=k}^{\infty} \tilde{E}_j\Big) \leqq \sum_{j=k}^{\infty} 2^{j(n-\alpha)} C_\alpha(E_j).$$

Since E is α-thin at 0 by assumption, the right-hand side tends to zero as $k \to \infty$ and then (8.2) follows.

THEOREM 8.1. *Let μ be a measure on $\mathbf{R}^n$ for which $|U_\alpha \mu| \not\equiv \infty$. Then there exists a set $E \subseteqq \mathbf{S}$ such that $C_\alpha(E) = 0$ and*

$$\lim_{r \to 0} U_\alpha \mu(r\xi) = U_\alpha \mu(0) \qquad \text{whenever } \xi \in \mathbf{S} - E. \tag{8.3}$$

PROOF. By Theorem 5.1, there exists a set E such that E is α-thin at 0 and

$$\lim_{x\to 0, x\in \mathbf{R}^n - E} U_\alpha\mu(x) = U_\alpha\mu(0). \tag{8.4}$$

As in Lemma 8.2, we consider the set

$$E^* = \bigcap_{k=1}^{\infty}\left(\bigcup_{j=k}^{\infty} \tilde{E}_j\right).$$

Then $C_\alpha(E^*) = 0$ by Lemma 8.2. Further, if $\xi \in \mathbf{S} - E^*$, then we can find k such that

$$\xi \notin \bigcup_{j=k}^{\infty} \tilde{E}_j,$$

that is,

$$r\xi \notin \bigcup_{j=k}^{\infty} E_j = E \cap B(0, 2^{-k+1}) \qquad \text{whenever } 0 < r < 2^{-k+1},$$

which together with (8.4) implies that (8.3) holds for this ξ.

In the same way we have by Theorem 5.3,

THEOREM 8.2. *Let μ be a measure on $\mathbf{R}^n$ for which $|U_\alpha\mu| \not\equiv \infty$. Then there exists a set $E \subsetneqq \mathbf{S}$ such that $C_\alpha(E) = 0$ and*

$$\lim_{r\to 0} [U_\alpha(r)]^{-1} U_\alpha\mu(r\xi) = \mu(\{0\}) \qquad \textit{whenever } \xi \in \mathbf{S} - E.$$

2.9 Energy integral

In case $\alpha < n$, for measures μ and ν, we define the mutual α-energy by the integral

$$\mathcal{E}_\alpha(\mu, \nu) = \int U_\alpha\mu(x)\, d\nu(x);$$

We call $\mathcal{E}_\alpha(\mu) = \mathcal{E}_\alpha(\mu, \mu)$ the α-energy of μ, and set

$$\mathcal{E}_\alpha = \{\mu : \mathcal{E}_\alpha(\mu) < \infty\}.$$

By the Riesz composition formula, we have the following inequality.

THEOREM 9.1. *For measures μ and ν,*

$$\mathcal{E}_\alpha(\mu, \nu) \leqq \sqrt{\mathcal{E}_\alpha(\mu)}\sqrt{\mathcal{E}_\alpha(\nu)}.$$

LEMMA 9.1. *Let $\{\mu_j\} \subseteqq \mathcal{E}_\alpha$. If $\mathcal{E}_\alpha(\mu_j)$ are bounded, then there exist $\mu_0 \in \mathcal{E}_\alpha$ and a subsequence $\{\mu_{j(k)}\}$ such that $\{\mu_{j(k)}\}$ converges vaguely to μ_0 and*

$$\lim_{k\to\infty} \mathcal{E}_\alpha(\mu_{j(k)}, \nu) = \mathcal{E}_\alpha(\mu_0, \nu) \qquad \text{for all } \nu \in \mathcal{E}_\alpha \ . \tag{9.1}$$

PROOF. In view of Corollary 1.2, $\{U_{\alpha/2}\mu_j\}$ is bounded in L^2. Hence, by Theorem 12.9 in Chapter 1, we can find a subsequence $\{\mu_{j(k)}\}$ such that $\{U_{\alpha/2}\mu_{j(k)}\}$ converges to a function f_0 weakly in L^2, that is,

$$\lim_{k\to\infty} \int U_{\alpha/2}\mu_{j(k)}\ g\ dx = \int f_0\ g\ dx \qquad \text{for all } g \in L^2.$$

For an open ball $B = B(a,r)$, let χ_B denote the characteristic function of B. Then we see that $U_\alpha\chi_B$ is continuous on $\mathbf{R}^n$ and vanishes at infinity, so that $\chi_B \in \mathcal{E}_\alpha$. Further, since $U_\alpha\chi_B(x) \geqq (2r)^{\alpha-n}|B| = c$ for $x \in B = B(a,r)$, we note that

$$\mu_j(B) \leqq c^{-1} \int U_\alpha\chi_B\ d\mu_j \leqq c^{-1}\sqrt{\mathcal{E}_\alpha(\chi_B)}\sqrt{\mathcal{E}_\alpha(\mu_j)}.$$

Thus $\{\mu_j(B)\}$ is bounded, so that we can find a subsequence $\{\mu_{j(k')}\}$ of $\{\mu_{j(k)}\}$ which converges vaguely to a measure μ_0. Let $\varphi \in C_0^\infty(\mathbf{R}^n)$. Setting $\psi = U_{\alpha/2}\varphi$, we show that

$$\lim_{k'\to\infty} \int \psi\ d\mu_{j(k')} = \int \psi\ d\mu_0 = \int U_{\alpha/2}\mu_0\ \varphi\ dy. \tag{9.2}$$

If f_N is a continuous function on $\mathbf{R}^n$ such that $f_N = 1$ on $B(0,N)$, $f_N = 0$ outside $B(0,2N)$ and $0 \leqq f_N \leqq 1$ on $\mathbf{R}^n$, then

$$\lim_{k'\to\infty} \int \psi\ f_N\ d\mu_{j(k')} = \int \psi\ f_N\ d\mu_0.$$

Since $|\psi(x)| \leqq M(1+|x|)^{\alpha/2-n}$, we see that $|\psi(1-f_N)| \leqq MU_{\alpha/2}\chi_{B(a,1)}$ on $\mathbf{R}^n$ whenever N is large enough, say $N \geqq N(a)$, so that

$$\int |\psi(1-f_N)|\ d\mu_j \leqq M \int U_{\alpha/2}\chi_{B(a,1)}\ d\mu_j = M \int_{B(a,1)} U_{\alpha/2}\mu_j\ dy.$$

Hence we have for large N,

$$\begin{aligned}
&\limsup_{k'\to\infty} \left| \int \psi\ d\mu_{j(k')} - \int \psi\ d\mu_0 \right| \\
\leqq\ & \limsup_{k'\to\infty} \int |\psi(1-f_N)|\ d\mu_{j(k')} + \int |\psi(1-f_N)|\ d\mu_0 \\
\leqq\ & M \limsup_{k'\to\infty} \left(\int_{B(a,1)} U_{\alpha/2}\mu_{j(k')}(y)\ dy + \int_{B(a,1)} U_{\alpha/2}\mu_0(y)\ dy \right) \\
\leqq\ & M \left(\int_{B(a,1)} f_0(y)\ dy + \int_{B(a,1)} U_{\alpha/2}\mu_0(y)\ dy \right) \\
\leqq\ & M \left\{ \left(\int_{B(a,1)} [f_0(y)]^2\ dy \right)^{1/2} + \left(\int_{B(a,1)} [U_{\alpha/2}\mu_0(y)]^2\ dy \right)^{1/2} \right\}.
\end{aligned}$$

Letting $|a| \to \infty$, we obtain (9.2). Now we find

$$\begin{aligned}\lim_{k'\to\infty} \int \psi \, d\mu_{j(k')} &= \lim_{k'\to\infty} \int U_{\alpha/2}\varphi \, d\mu_{j(k')} \\ &= \lim_{k'\to\infty} \int U_{\alpha/2}\mu_{j(k')} \, \varphi \, dy = \int f_0 \, \varphi \, dy.\end{aligned}$$

Hence it follows that $U_{\alpha/2}\mu_0 = f_0$. If we note that

$$\begin{aligned}\lim_{k'\to\infty} \mathcal{E}_\alpha(\mu_{j(k')}, \nu) &= a(\alpha/2, \alpha/2) \lim_{k'\to\infty} \int U_{\alpha/2}\mu_{j(k')}(y) U_{\alpha/2}\nu(y) \, dy \\ &= a(\alpha/2, \alpha/2) \int f_0(y) U_{\alpha/2}\nu(y) \, dy \\ &= a(\alpha/2, \alpha/2) \int U_{\alpha/2}\mu_0(y) U_{\alpha/2}\nu(y) \, dy \\ &= \mathcal{E}_\alpha(\mu_0, \nu)\end{aligned}$$

for $\nu \in \mathcal{E}_\alpha$, then (9.1) holds, and the proof is completed.

Denote by $\mathcal{E}_\alpha$ be the family of all measures μ with finite α-energy, and by $\overline{\mathcal{E}}_\alpha$ the space of all differences of measures in $\mathcal{E}_\alpha$. If $\lambda = \mu - \nu$ for μ, $\nu \in \mathcal{E}_\alpha$, then define

$$\begin{aligned}\overline{\mathcal{E}}_\alpha(\mu - \nu) &= \overline{\mathcal{E}}_\alpha(\mu - \nu, \mu - \nu) \\ &= \mathcal{E}_\alpha(\mu, \mu) - 2\mathcal{E}_\alpha(\mu, \nu) + \mathcal{E}_\alpha(\nu, \nu) \\ &= a(\alpha/2, \alpha/2) \int [U_{\alpha/2}\mu - U_{\alpha/2}\nu]^2 \, dx.\end{aligned}$$

In view of Theorem 9.1, $\overline{\mathcal{E}}_\alpha(\cdot)$ is a norm in $\overline{\mathcal{E}}_\alpha$. In fact we have

LEMMA 9.2. *For μ, $\nu \in \mathcal{E}_\alpha$,*

$$\left| \sqrt{\mathcal{E}_\alpha(\mu)} - \sqrt{\mathcal{E}_\alpha(\nu)} \right| \leqq \sqrt{\overline{\mathcal{E}}_\alpha(\mu - \nu)}. \tag{9.3}$$

LEMMA 9.3. *The mutual α-energy $\overline{\mathcal{E}}_\alpha(\cdot, \cdot)$ defines an inner product in the space $\overline{\mathcal{E}}_\alpha$.*

In fact, $\overline{\mathcal{E}}_\alpha(\mu - \nu) \geqq 0$, and $\overline{\mathcal{E}}_\alpha(\mu - \nu) = 0$ if and only if $\mu = \nu$, with the aid of Corollary 2.5. Further, if λ_1, $\lambda_2 \in \overline{\mathcal{E}}_\alpha$, then

$$|\overline{\mathcal{E}}_\alpha(\lambda_1, \lambda_2)| \leqq \sqrt{\overline{\mathcal{E}}_\alpha(\lambda_1)} \sqrt{\overline{\mathcal{E}}_\alpha(\lambda_2)}. \tag{9.4}$$

THEOREM 9.2. *Let $\mu_j \in \mathcal{E}_\alpha$ for each j. If $\{\mu_j\}$ is a Cauchy sequence in $\overline{\mathcal{E}}_\alpha$, that is, $\lim_{j,k\to\infty} \overline{\mathcal{E}}_\alpha(\mu_j - \mu_k) = 0$, then there exists $\mu_0 \in \mathcal{E}_\alpha$ for which*

$$\lim_{j\to\infty} \overline{\mathcal{E}}_\alpha(\mu_j - \mu_0) = 0.$$

PROOF. By Lemma 9.2, $\{\mathcal{E}_\alpha(\mu_j)\}$ is a Cauchy sequence, so that by Lemma 9.1, there exists $\mu_0 \in \mathcal{E}_\alpha$ such that

$$\lim_{j\to\infty} \overline{\mathcal{E}}_\alpha(\mu_j - \mu_0, \nu) = 0$$

for any $\nu \in \mathcal{E}_\alpha$. We have

$$\begin{aligned} \overline{\mathcal{E}}_\alpha(\mu_j - \mu_0) &= \lim_{k\to\infty} \overline{\mathcal{E}}_\alpha(\mu_j - \mu_k, \mu_j - \mu_0) \\ &\leqq \limsup_{k\to\infty} \sqrt{\overline{\mathcal{E}}_\alpha(\mu_j - \mu_k)}\sqrt{\overline{\mathcal{E}}_\alpha(\mu_j - \mu_0)} \\ &\leqq \limsup_{k\to\infty} \sqrt{\overline{\mathcal{E}}_\alpha(\mu_j - \mu_k)}\{\sqrt{\mathcal{E}_\alpha(\mu_j)} + \sqrt{\mathcal{E}_\alpha(\mu_0)}\}, \end{aligned}$$

which tends to zero as $j \to \infty$, or

$$\lim_{j\to\infty} \overline{\mathcal{E}}_\alpha(\mu_j - \mu_0) = 0.$$

REMARK 9.1. The space $\overline{\mathcal{E}}_\alpha$ is not complete with respect to the norm $\overline{\mathcal{E}}_\alpha$. In fact, letting $\mu_r = \dfrac{1}{|B(0,r)|}\chi_{B(0,r)}$, we note that

$$U_\alpha\mu_r(x) = r^\alpha (U_\alpha\mu_1)(x/r).$$

Hence $U_\alpha\mu_r$ converges to $U_\alpha\mu_1$ uniformly on $\mathbf{R}^n$ as $r \to 1$. Choose $r(j)$ so that $r(j) \downarrow 1$ and $|U_\alpha\mu_{r(j)} - U_\alpha\mu_1| \leqq 2^{-j}$ on $\mathbf{R}^n$. Now define

$$\lambda_k = k\mu_1 - \sum_{j=1}^{k} \mu_{r(j)}$$

for each positive integer k. Then

$$\begin{aligned} \overline{\mathcal{E}}_\alpha(\lambda_k - \lambda_{k+1}) &= \int [U_\alpha\mu_1 - U_\alpha\mu_{r(k+1)}]\, d(\mu_1 - \mu_{j(k+1)}) \\ &\leqq \int |U_\alpha\mu_1 - U_\alpha\mu_{r(k+1)}|\, d(\mu_1 + \mu_{j(k+1)}) \leqq 2^{-k}, \end{aligned}$$

which implies that $\{\overline{\mathcal{E}}_\alpha(\lambda_k)\}$ is a Cauchy sequence. Suppose $\{\overline{\mathcal{E}}_\alpha(\lambda_k - \lambda)\} \to 0$ for some signed measure λ. Then we see that for any $\varphi \in C_0^\infty(\mathbf{R}^n)$,

$$\lim_{k\to\infty} \int \varphi\, d\lambda_k = \int \varphi\, d\lambda$$

with the aid of Theorem 2.9. On the other hand, if $\varphi_m \in C_0^\infty(\mathbf{R}^n)$, $\varphi_m = 1$ on $B(0,2) - B(0,r(m))$ and $\varphi_m = 0$ on $\overline{B(0,r(m+1))} \cup (\mathbf{R}^n - B(0,3))$, then

$$\int \varphi_m\, d\lambda = \lim_{k\to\infty} \int \varphi_m\, d\lambda_k = -m.$$

On the other hand,

$$\left|\int \varphi_m\, d\lambda\right| \leqq \int_{B(0,3)} d|\lambda| < \infty,$$

which yields a contradiction.

2.10 Gauss variation

If a property holds on G except for a set of C_α-capacity zero, then we say that the property holds α-q.e. on G.

Throughout this section, let $\alpha < n$.

THEOREM 10.1. *If K is a compact set in $\mathbf{R}^n$ with $C_\alpha(K) > 0$, then there exists a measure $\mu_K \in \mathcal{E}_\alpha(K) = \mathcal{E}_\alpha \cap \mathcal{M}(K)$ such that*

(i) $U_\alpha \mu_K(x) \geqq 1$ *α-q.e. on K;*

(ii) $U_\alpha \mu_K(x) \leqq 1$ *for any $x \in S_{\mu_K}$.*

PROOF. Denoting by $\mathcal{M}_1(K)$ the family of all unit measures in $\mathcal{M}(K)$, we set

$$\mathcal{E}_{\alpha,1}(K) = \mathcal{E}_\alpha \cap \mathcal{M}_1(K).$$

Now consider the quantity

$$A = \inf\{\mathcal{E}_\alpha(\mu) : \mu \in \mathcal{E}_{\alpha,1}(K)\}.$$

Since $c_\alpha(K) = C_\alpha(K) > 0$ with the aid of Theorem 4.3, $\mathcal{E}_{\alpha,1}(K)$ is seen to be non-empty, so that A is finite. Hence there exists a sequence $\{\mu_j\} \subseteqq \mathcal{E}_{\alpha,1}(K)$ such that $\{\mu_j\}$ converges vaguely to μ_0 and

$$A = \lim_{j \to \infty} \mathcal{E}_\alpha(\mu_j).$$

Noting that

$$A \leqq \mathcal{E}_\alpha((\mu_j + \mu_k)/2) = 2^{-1}\mathcal{E}_\alpha(\mu_j) + 2^{-1}\mathcal{E}_\alpha(\mu_k) - 4^{-1}\overline{\mathcal{E}}_\alpha(\mu_j - \mu_k),$$

we insist that

$$\overline{\mathcal{E}}_\alpha(\mu_j - \mu_k) \to 0 \qquad \text{as } j,\, k \to \infty.$$

Thus, in view of Theorem 9.2, we see that $\mu_0 \in \mathcal{E}_\alpha$ and

$$\lim_{j \to \infty} \overline{\mathcal{E}}_\alpha(\mu_j - \mu_0) = 0.$$

Further note that μ_0 is supported by K, $\mu_0(K) = 1$ and

$$A = \mathcal{E}_\alpha(\mu_0) > 0. \tag{10.1}$$

On the other hand, if $0 < t < 1$ and $\mu \in \mathcal{E}_{\alpha,1}(K)$, then $(1-t)\mu_j + t\mu \in \mathcal{M}_1(K)$, so that

$$A \leqq \mathcal{E}_\alpha((1-t)\mu_j + t\mu).$$

This proves, by letting $j \to \infty$, that

$$-2t\overline{\mathcal{E}}_\alpha(\mu_0, \mu_0 - \mu) + t^2\overline{\mathcal{E}}_\alpha(\mu_0 - \mu) \geqq 0.$$

By dividing both sides by t and then letting $t \to 0$, we obtain

$$\overline{\mathcal{E}}_\alpha(\mu_0, \mu_0 - \mu) \leqq 0,$$

or

$$A = \mathcal{E}_\alpha(\mu_0, \mu_0) \leqq \mathcal{E}_\alpha(\mu_0, \mu). \tag{10.2}$$

From (10.2), we can show that $U_\alpha\mu_0(x) \geqq A$ on K except for a set of c_α-capacity zero, so that

$$U_\alpha\mu_0(x) \geqq A \qquad \alpha\text{-q.e. on } K.$$

Suppose $U_\alpha\mu_0(x_0) > A$ for $x_0 \in S_{\mu_0}$. By the lower semicontinuity, we can find a ball $B = B(x_0, r_0)$ for which we see that

$$U_\alpha\mu_0(x) > A \qquad \text{whenever } x \in B\ .$$

Taking $\psi \in C_0^\infty(B)$ such that $0 \leqq \psi \leqq 1$ on $\mathbf{R}^n$ and $\psi(x_0) > 0$, we define $\nu_0(t) = (1 - t\psi)\mu_0$ for $|t| < 1$ and $a_0(t) = \nu_0(t)(\mathbf{R}^n)$. Then, considering $\nu_0(t)/a_0(t) \in \mathcal{E}_{\alpha,1}(K)$, we have by (10.2)

$$A \leqq \mathcal{E}_\alpha(\mu_0, \nu_0(t)/a_0(t)).$$

Consequently,

$$A \leqq \mathcal{E}_\alpha(\mu_0, \nu_0(t)/a_0(t)) = A/a_0(t) - t/a_0(t) \int U_\alpha\mu_0\ \psi\ d\mu_0.$$

Letting $t \uparrow 0$ and $t \downarrow 0$, we have

$$\int U_\alpha\mu_0\ \psi\ d\mu_0 = A \int \psi\ d\mu_0,$$

which follows a contradiction. Thus

$$U_\alpha\mu_0(x_0) \leqq A$$

as required. Finally, in view of (10.1), we see that

$$\mu_0(\{x \in S_{\mu_0} : U_\alpha\mu_0(x_0) < A\}) = 0.$$

Thus $\mu_K = \mu_0/A$ satisfies all the required conditions.

Remark 10.1. As seen above,

$$C_\alpha(K) \leqq \mu_K(K) \leqq 2^{n-\alpha}C_\alpha(K).$$

In case $\alpha \leqq 2$, it will be shown, by applying maximum principle for subharmonic functions, that

$$\mu_K(K) = C_\alpha(K) = [\inf\{\mathcal{E}_\alpha(\mu) : \mu \in \mathcal{E}_{\alpha,1}(K)\}]^{-1}.$$

THEOREM 10.2. *If G is a non-empty open set in $\mathbf{R}^n$ such that $C_\alpha(G) < \infty$, then there exists a measure $\mu_G \in \mathcal{E}_\alpha$ supported by $\overline{G}$ such that*

(i) $U_\alpha \mu_G(x) \geqq 1$ *for any* $x \in G$;

(ii) $U_\alpha \mu_G(x) \leqq 1$ *for any* $x \in S_{\mu_G}$;

(iii) $\mu_G(\{x : U_\alpha \mu_G(x) < 1\}) = 0$.

In fact, take a sequence $\{K_j\}$ of compact sets in G such that

$$K_j \subsetneqq \operatorname{Int}(K_{j+1}) \quad \text{and} \quad \bigcup_{j=1}^{\infty} K_j = G.$$

For each j, choose $\mu_j = \mu_{K_j}$ as in Theorem 10.1. In view of Remark 10.1,

$$\mathcal{E}_\alpha(\mu_j) = \mu_j(K_j) \leqq 2^{n-\alpha} C_\alpha(K_j) \leqq 2^{n-\alpha} C_\alpha(G),$$

and thus $\{\mathcal{E}_\alpha(\mu_j)\}$ is bounded. In view of Lemma 9.1, we may assume that $\{\mu_j\}$ converges vaguely to a measure $\mu_0 \in \mathcal{E}_\alpha$. Since $\{\mathcal{E}_\alpha(\mu_j)\}$ is nondecreasing, Theorem 9.2 implies that

$$\lim_{j\to\infty} \overline{\mathcal{E}}_\alpha(\mu_j - \mu_0) = 0.$$

As in the proof of (9.2), we have

$$1 \leqq \mathcal{E}_\alpha(\mu_0, \nu) \qquad \text{whenever } \nu \in \mathcal{E}_{\alpha,1}(K_j).$$

Now applying this with $\xi_{\overline{B}} = |\overline{B}|^{-1}\chi_{\overline{B}}$ for a closed ball $\overline{B}$ in G, we see that μ_0 satisfies (i). Assertion (ii) can be proved in the same way as in the proof of Theorem 10.1, and then μ_0 satisfies (iii).

THEOREM 10.3 (Gauss-Frostman). *Let K be a compact set in $\mathbf{R}^n$ and f be a positive continuous function on K. Then there exists a measure $\mu_{f,K} \in \mathcal{E}_\alpha(K)$ such that*

(i) $U_\alpha \mu_{f,K} \geqq f$ *α-q.e. on* K;

(ii) $U_\alpha \mu_{f,K} \leqq f$ *on* $S_{\mu_{f,K}}$.

PROOF. If $C_\alpha(K) = 0$, then we may take $\mu = 0$; note here that $C_\alpha(K) = 0$ if and only if $\mathcal{E}_\alpha(K) = \{0\}$. Hence we may assume that $C_\alpha(K) > 0$. Consider the Gauss integral

$$V(\mu) = \mathcal{E}_\alpha(\mu) - 2\int f\, d\mu$$

and consider the quantity

$$A = \inf\{V(\mu) : \mu \in \mathcal{E}_\alpha(K)\}.$$

Clearly, $A \leqq 0$. Further, by Theorem 10.1, we have for $\mu \in \mathcal{E}_\alpha(K)$

$$\int f\, d\mu \leqq (\max f)\mathcal{E}_\alpha(\mu_K, \mu) \leqq (\max f)\sqrt{\mathcal{E}_\alpha(\mu_K)}\sqrt{\mathcal{E}_\alpha(\mu)},$$

from which it follows that

$$A \geqq -(\max f)^2 \mathcal{E}_\alpha(\mu_K).$$

Thus A is finite. Hence there exists a sequence $\{\mu_j\} \subseteqq \mathcal{E}_\alpha(K)$ such that

$$A = \lim_{j\to\infty} V(\mu_j).$$

Since $\{\mathcal{E}_\alpha(\mu_j)\}$ is bounded, by Lemma 9.1 we may assume the existence of $\mu_0 \in \mathcal{E}_\alpha(K)$ such that

$$\lim_{j\to\infty} \mathcal{E}_\alpha(\mu_j, \nu) = \mathcal{E}_\alpha(\mu_0, \nu)$$

for any $\nu \in \mathcal{E}_\alpha$. Noting that

$$A \leqq [V(\mu_j) + V(\mu_k)]/2 - 4^{-1}\overline{\mathcal{E}}_\alpha(\mu_j - \mu_k),$$

we insist that $\{\mu_j\}$ is a Cauchy sequence and

$$\lim_{j\to\infty} \overline{\mathcal{E}}_\alpha(\mu_j - \mu_0) = 0.$$

Further note that

$$A = V(\mu_0). \tag{10.3}$$

On the other hand, if $0 < t < 1$ and $\nu \in \mathcal{E}_\alpha(K)$, then $\mu_0 + t\nu \in \mathcal{E}_\alpha(K)$, so that

$$\mathcal{E}_\alpha(\mu_0, \nu) \geqq \int f\, d\nu. \tag{10.4}$$

Similarly, if $0 < t < 1$, then $(1-t)\mu_0 \in \mathcal{E}_\alpha(K)$, which yields

$$\int U_\alpha \mu_0\, d\mu_0 = \int f\, d\mu_0. \tag{10.5}$$

In view of (10.4), we see that

$$U_\alpha \mu_0 \geqq f \qquad \alpha\text{-q.e. on } K.$$

Further, it follows from (10.5) that

$$U_\alpha \mu_0 \leqq f \qquad \text{on } S_{\mu_0}.$$

COROLLARY 10.1. *Let F be a closed set in $\mathbf{R}^n$. If $\mu \in \mathcal{E}_\alpha$, then there exists a unique measure $\mu_F \in \mathcal{E}_\alpha(F) = \mathcal{E}_\alpha \cap \mathcal{M}(F)$ such that*

(i) $U_\alpha \mu_F \geqq U_\alpha \mu$ *α-q.e. on F;*

(ii) $U_\alpha \mu_F \leqq U_\alpha \mu$ *on S_{μ_F}.*

PROOF. The above proof is applicable to the case $f = U_\alpha \mu$ and hence we can find $\mu_0 \in \mathcal{E}_\alpha(F)$ such that

$$\mathcal{E}_\alpha(\mu_0, \nu) \geqq \mathcal{E}_\alpha(\mu, \nu) \qquad \text{whenever } \nu \in \mathcal{E}_\alpha(F) \tag{10.6}$$

and

$$\mathcal{E}_\alpha(\mu_0, \mu_0) = \mathcal{E}_\alpha(\mu, \mu_0). \tag{10.7}$$

In view of these properties, we see that μ_0 has the required properties. To show the uniqueness, let $\mu_1 \in \mathcal{E}_\alpha(F)$ satisfy (i) and (ii). Then, replacing ν by μ_1 in (10.6), we have

$$\mathcal{E}_\alpha(\mu_0, \mu_1) \geqq \mathcal{E}_\alpha(\mu, \mu_1) = \mathcal{E}_\alpha(\mu_1, \mu_1),$$

and, conversely,

$$\mathcal{E}_\alpha(\mu_1, \mu_0) \geqq \mathcal{E}_\alpha(\mu, \mu_0) = \mathcal{E}_\alpha(\mu_0, \mu_0).$$

Hence it follows that

$$\overline{\mathcal{E}}_\alpha(\mu_0 - \mu_1) \leqq 0,$$

so that $\mu_1 = \mu_0$.

In the logarithmic case, we show the following results.

THEOREM 10.4. *If K is a compact set in $\mathbf{B}$ with $C_n(K) > 0$, then there exist a number A and a measure $\mu \in \mathcal{E}_{n,1}(K)$ such that*

(i) $U_n \mu \geqq A$ *n-q.e. on K;*

(ii) $U_n \mu \leqq A$ *on S_μ.*

PROOF. As in the proof of Theorem 10.1, consider the quantity

$$L = \inf\left\{ \int \left(\int \log(2/|x-y|)\, d\mu(y) \right) d\mu(x) : \mu \in \mathcal{E}_{n,1}(K) \right\}.$$

Since $C_n(K) > 0$, we see that L is finite. Hence, in the same way as in the proof of Theorem 10.1, we find $\mu_0 \in \mathcal{E}_{n,1}(K)$ such that

$$\int \log(2/|x-y|)\, d\mu_0(x) \geqq L \qquad \text{for } n\text{-q.e. } x \in K,$$

$$\int \log(2/|x-y|)\, d\mu_0(y) \leqq L \qquad \text{for any } x \in S_{\mu_0}$$

and

$$\mu_0\left(\left\{x : \int \log(2/|x-y|)\, d\mu_0(y) < L\right\}\right) = 0.$$

Now we may take $A = L - \log 2$.

THEOREM 10.5. *Let K be a compact set in $\mathbf{R}^n$ and f be a positive continuous function on K. If $C_n(K) > 0$, then there exist a measure $\mu \in \mathcal{E}_{n,1}(K)$ and a number γ such that*

(i) $U_n\mu \geqq f + \gamma$ *n-q.e. on K;*

(ii) $U_n\mu \leqq f + \gamma$ *on S_μ.*

For this purpose, let

$$V(\mu) = \int U_n\mu(x)\, d\mu(x) - 2\int f\, d\mu$$

and consider the quantity

$$A = \inf\{V(\mu) : \mu \in \mathcal{E}_{n,1}(K)\}.$$

The remaining part of the proof is similar to that of Theorem 10.3.

Chapter 3

Harmonic functions

In the planar domains, real and imaginary parts of holomorphic functions are harmonic by Cauchy-Riemann equations. Harmonic functions are characterized by the mean-value property, and superharmonic functions satisfy the super-mean-value property. The goal is the study of Dirichlet problem. In this chapter we deal with the classical Dirichlet problem for the Laplace operator.

3.1 Harmonic functions

A locally integrable function u on G is called harmonic in G if it has the following mean-value property:

$$u(x) = \frac{1}{|B|}\int_B u(y)\ dy$$

for any ball $B = B(x,r)$ with closure in G. It is easy to see that harmonic functions are continuous.

THEOREM 1.1. *Let $u \in L^1_{loc}(G)$. Then the following are equivalent.*

(1) *u is harmonic in G.*

(2) *$u \in C(G)$ and*

$$u(x) = \frac{1}{|S(x,r)|}\int_{S(x,r)} u(y)\ dS(y) \qquad \text{whenever } \overline{B(x,r)} \subsetneqq G.$$

(3) *$u \in C^2(G)$ and $\Delta u = 0$ in G.*

(4) *$u \in C(G)$ and $\displaystyle\int u(x)\Delta\varphi(x)\ dx = 0$ for any $\varphi \in C_0^\infty(G)$.*

PROOF. (1) $\Rightarrow$ (3) : Let ψ be a function in $C_0^\infty(\mathbf{B})$ such that ψ is radial and $\displaystyle\int \psi\ dx = 1$. For $\delta > 0$, set

$$\psi_\delta(x) = \delta^{-n}\psi(x/\delta)$$

and consider the function defined by the convolution

$$u * \psi_\delta(x) = \int u(x-y)\psi_\delta(y)\,dy,$$

which is well defined in $G_\delta = \{x \in G : \mathrm{dist}(x, \partial G) > \delta\}$. If $x \in G_\delta$, then we have by Corollary 7.2 in Chapter 1

$$\begin{aligned} u * \psi_\delta(x) &= \int u(x-y)\psi_\delta(y)\,dy \\ &= -\int \left(\int_{B(0,r)} u(x-y)\,dy\right)\psi'_\delta(r)\,dr \\ &= -\int (\sigma_n r^n u(x))\psi'_\delta(r)\,dr \\ &= \sigma_n u(x)\int \psi_\delta(r)\,d(r^n) \\ &= u(x)\int \psi_\delta(y)\,dy = u(x). \end{aligned}$$

It thus follows that $u \in C^2(G)$. The above considerations also imply

$$\begin{aligned} \Delta(u * \psi_\delta)(x) &= (u * \Delta\psi_\delta)(x) \\ &= u(x)\int \Delta\psi_\delta(y)\,dy = 0 \end{aligned}$$

on G_δ, which shows that $\Delta u = 0$ in G. Hence (1) implies (3).

(3) $\Rightarrow$ (2) : For any fixed $x \in G$, define

$$N_x(y) = U_2(x-y) = \begin{cases} |x-y|^{2-n} & \text{in case } n \geqq 3, \\ \log(1/|x-y|) & \text{in case } n = 2. \end{cases}$$

Then easy computations imply that $\Delta N_x = 0$ in $\mathbf{R}^n - \{x\}$. Applying Green's formula in the ring domain $R(a,b) = \{y : a < |x-y| < b\}$, we have

$$0 = \int_{R(a,b)} \{(\Delta u)N_x - u(\Delta N_x)\}\,dy = \int_{\partial R(a,b)} \{u_{\mathbf{n}} N_x - u(N_x)_{\mathbf{n}}\}\,dS(y),$$

where $u_{\mathbf{n}}$ denotes the normal derivative of u on the boundary. Here we note that in case $n \geqq 3$,

$$\begin{aligned} \int_{S(x,r)} u_{\mathbf{n}} N_x\,dS &= r^{2-n}\int_{S(x,r)} u_{\mathbf{n}}\,dS \\ &= r^{2-n}\int_{S(x,r)} \Delta u\,dy = 0 \end{aligned}$$

and

$$\int_{S(x,r)} u(N_x)_{\mathbf{n}}\,dS = (2-n)r^{1-n}\int_{S(x,r)} u\,dS.$$

Thus we have

$$b^{1-n}\int_{S(x,b)} u\ dS = a^{1-n}\int_{S(x,a)} u\ dS.$$

The right-hand side tends to $\omega_n u(x)$ as $a \to 0$, and hence

$$u(x) = \frac{1}{|S(x,b)|}\int_{S(x,b)} u\ dS.$$

The case $n = 2$ is treated similarly, and hence (2) follows.

Applying polar coordinates about x, we see that (1) follows from (2). Thus (1), (2) and (3) are equivalent to each other.

Since the implication (3) $\Rightarrow$ (4) is trivial, we have only to show that (4) implies one of (1) $\sim$ (3). In fact we show that (4) implies (1).

(4) $\Rightarrow$ (1) : As in the first part of the proof, we consider the convolution $u * \psi_\delta$. If (4) holds, then

$$\Delta(u * \psi_\delta)(x) = \int u(y)\Delta\psi_\delta(x-y)\ dy = 0$$

for $x \in G_\delta$, so that $u * \psi_\delta$ is harmonic in G_δ in view of the implication (3) $\Rightarrow$ (1). Since $u * \psi_\delta$ is convergent to u locally uniformly in G as $\delta \to 0$, $u * \psi_\delta$ and then u have the mean-value property. Thus (4) implies (1), and the proof is completed.

REMARK 1.1. Since $\Delta N_x = 0$ on $\mathbf{R}^n - \{x\}$, N_x is harmonic in $\mathbf{R}^n - \{x\}$ by the implication (3) $\Rightarrow$ (1). The function $N(y) = N_0(y)$ is called the fundamental solution for the Laplacian, that is,

$$\Delta N = -a_n\omega_n\delta_0$$

in the sense of distributions, where δ_0 denotes the Dirac measure at the origin and

$$a_n = \begin{cases} n-2 & n \geqq 3, \\ 1 & n = 2. \end{cases}$$

In fact, if $\varphi \in C_0^\infty$, then Green's formula implies that

$$\varphi(x) = -\frac{1}{a_n\omega_n}\int N(x-y)\Delta\varphi(y)\ dy. \tag{1.1}$$

REMARK 1.2. If T is a distribution on G such that $\Delta T = 0$ on G, that is, T is a harmonic distribution on G, then the above proof shows that $T * \psi_\delta$ is harmonic in G_δ and hence

$$(T * \psi_\delta) * \psi_{\delta'} = T * \psi_\delta$$

in $G_{\delta+\delta'}$. By letting $\delta \to 0$, we see that

$$T = T * \psi_{\delta'}$$

on $G_{\delta'}$, so that T is considered as a usual harmonic function on G. This fact is known as Weyl's lemma.

The classical Dirichlet problem is that of finding a harmonic function u on G such that

$$u(x) = f(x) \qquad \text{for all } x \in \partial G \tag{1.2}$$

for a function f defined on the boundary. Here (1.2) is understood as

$$\overline{u}(x) \equiv \lim_{y \to x, y \in G} u(y) = f(x) \qquad \text{for all } x \in \partial G. \tag{1.3}$$

If a solution u exists for a finite-valued initial data f, then f is continuous on ∂G (see Theorem 6.1 given later). Further, by minimum and maximum principles (see Corollary 2.2 given later), a solution is unique if it exists.

Green's formula gives a good tool to solve the Dirichlet problem.

THEOREM 1.2. *Let D be a bounded domain for which Green's formula holds. If $u \in C^1(\overline{D})$ and u is harmonic in D, then*

$$u(x) = \frac{1}{a_n \omega_n} \int_{\partial D} \{u_{\mathbf{n}} N_x - u(N_x)_{\mathbf{n}}\}\, dS(y) \tag{1.4}$$

for every $x \in D$.

Let $B = B(a, R)$ be a ball. For x, consider the inversion of x with respect to $S(a, R)$:

$$x^* = a + R^2 \frac{x - a}{|x - a|^2}.$$

If $x \in B(a, R)$ and $y \in S(a, R)$, then we see that

$$|x^* - y| = |x^* - a||x - y|/R.$$

Since $N_{x^*}(y)$ is harmonic in $B(a, R)$, we have

$$\int_{\partial D} \{u_{\mathbf{n}} N_{x^*} - u(N_{x^*})_{\mathbf{n}}\}\, dS(y) = 0 \tag{1.5}$$

in $B(a, R)$. Consider the function

$$G_x(y) = N_x(y) - \left(\frac{|x^* - a|}{R}\right)^{n-2} N_{x^*}(y).$$

If $u \in C^1(\overline{B(a, R)})$ and u is harmonic in $B(a, R)$, then we have by (1.4) and (1.5)

$$u(x) = -\frac{1}{a_n \omega_n} \int_{S(a,R)} u(y)(G_x)_{\mathbf{n}}(y)\, dS(y).$$

Computing the normal derivative $(G_x)_{\mathbf{n}}$ on $S(a,R)$, we have the following result.

THEOREM 1.3. *If u is harmonic in $B(a,R)$ and u is continuous on $\overline{B(a,R)}$, then*

$$u(x) = \frac{1}{\omega_n R} \int_{S(a,R)} \frac{R^2 - |x-a|^2}{|x-y|^n}\, u(y)\; dS(y)$$

for every $x \in B(a,R)$.

In fact, if $0 < r < R$, then

$$u(x) = \frac{1}{\omega_n r} \int_{S(a,r)} \frac{r^2 - |x-a|^2}{|x-y|^n}\, u(y)\; dS(y)$$

for every $x \in B(a,r)$, and hence the required result follows by letting $r \to R$.

We say that

$$P(x,y) = P_B(x,y) = \frac{1}{\omega_n R} \frac{R^2 - |x-a|^2}{|x-y|^n}$$

is the Poisson kernel for $B = B(a,R)$, which has the following properties :

(P1) For $y \in S(a,R)$, $P(\cdot, y)$ is harmonic in $B(a,R)$.

(P2) For $y \in S(a,R)$, $P(\cdot, y)$ vanishes on $S(a,R) - \{y\}$.

(P3) For $x \in B(a,R)$,

$$\int_{S(a,R)} P(x,y)\; dS(y) = 1.$$

The last assertion can be obtained by considering $u \equiv 1$.

THEOREM 1.4. *If f is continuous on $S(a,R)$, then*

$$u(x) = \int_{S(a,R)} P(x,y) f(y)\; dS(y)$$

is the Dirichlet solution in $B(a,R)$ for f.

PROOF. It is easy to see that u is harmonic in $B(a,R)$. Since f is continuous on $S(a,r)$, f is bounded, that is,

$$|f| \leqq M < \infty \qquad \text{on } S(a,R).$$

Let $x_0 \in S(a,R)$. Given $\varepsilon > 0$, find $\delta > 0$ such that

$$|f(y) - f(x_0)| < \varepsilon \qquad \text{whenever } |y - x_0| < \delta.$$

Then (P2) gives

$$\begin{aligned}
&\int_{S(a,R)-B(x_0,\delta)} P(x,y)|f(y)-f(x_0)|\ dS(y) \\
\leqq\ & 2M\int_{S(a,R)-B(x_0,\delta)} P(x,y)\ dS(y) \to 0 \qquad \text{as } x\to x_0,\ x\in B(a,R).
\end{aligned}$$

Hence we have by (P3),

$$\begin{aligned}
& \limsup_{x\to x_0, x\in B(a,R)} |u(x)-f(x_0)| \\
\leqq\ & \limsup_{x\to x_0, x\in B(a,R)} \int_{S(a,R)\cap B(x_0,\delta)} P(x,y)|f(y)-f(x_0)|\ dS(y) \\
\leqq\ & \varepsilon \limsup_{x\to x_0, x\in B(a,R)} \int_{S(a,R)} P(x,y)\ dS(y) = \varepsilon,
\end{aligned}$$

which implies that

$$\lim_{x\to x_0, x\in B(a,R)} |u(x)-f(x_0)| = 0$$

as required.

THEOREM 1.5. *Let u be a harmonic function on $B(a,R)$. If $u \geqq 0$ on $B(a,R)$, then there exists a measure μ supported by $S(a,M)$ for which*

$$u(x) = \int_{S(a,R)} P(x,y)\ d\mu(y) \tag{1.6}$$

whenever $x \in B(a,R)$.

PROOF. In view of Theorem 1.3, if $0 < r < R$, then

$$u(x) = \int_{S(a,r)} P_{B(a,r)}(x,y)u(y)\ dS(y)$$

whenever $x \in B(a,r)$. Note, in view of the mean-value property, that

$$\int_{S(a,r)} u(y)\ dS(y) = |S(a,r)|u(a),$$

which is bounded. Hence there exist a measure μ and a sequence $\{r_j\}$ such that $r_j \uparrow R$ and the measure $d\mu_j = u\ dS|_{B(a,r_j)}$ converges vaguely to μ as $j\to\infty$. If $x \in B(a,R)$ is fixed, then $P_{B(a,r_j)}(x,\cdot)$ is uniformly convergent to $P_{B(a,R)}(x,\cdot)$ near $S(a,R)$, so that

$$\begin{aligned}
u(x) &= \int P_{B(a,r_j)}(x,y)\ d\mu_j(y) \\
&\to \int P_{B(a,R)}(x,y)\ d\mu(y) \qquad \text{as } j\to\infty.
\end{aligned}$$

Thus (1.6) holds, and the theorem is obtained.

As applications of Poisson integral representation, we can show the following Harnack inequality for harmonic functions.

THEOREM 1.6 (Harnack's inequality). *If u is a positive harmonic function on the ball $B(a,R)$, then*

$$\left(\frac{R}{R+r}\right)^{n-2}\frac{R-r}{R+r}\,u(a)\leqq u(x)\leqq\left(\frac{R}{R-r}\right)^{n-2}\frac{R+r}{R-r}\,u(a)$$

whenever $r=|x-a|<R$.

PROOF. Let $0<r<R'<R$. By Theorem 1.3, we see that

$$u(x)=\frac{1}{\sigma_n R'}\int_{B(0,R')}\frac{R'^2-r^2}{|x-y|^n}\,u(y)\,dS(y).$$

Note here that

$$\frac{R'^2-r^2}{(R'+r)^n}\leqq\frac{R'^2-r^2}{|x-y|^n}\leqq\frac{R'^2-r^2}{(R'-r)^n}$$

for $x\in S(a,r)$ and $y\in S(a,R')$. Since u is positive on $B(a,R)$, we have

$$\frac{R'^2-r^2}{(R'+r)^n\omega_n R'}\int_{S(a,R')}u(y)\,dS(y)\leqq u(x)\leqq\frac{R'^2-r^2}{(R'-r)^n\omega_n R'}\int_{S(a,R')}u(y)\,dS(y).$$

By mean-value property, we find

$$\left(\frac{R'}{R'+r}\right)^{n-2}\frac{R'-r}{R'+r}\,u(a)\leqq u(x)\leqq\left(\frac{R'}{R'-r}\right)^{n-2}\frac{R'+r}{R'-r}\,u(a),$$

which yields the required inequalities by letting $R'\uparrow R$.

THEOREM 1.7 (Harnack's principle). *If $\{u_j\}$ is an increasing sequence of harmonic functions on a domain D, then it converges to a harmonic function locally uniformly in D if $\lim_{j\to\infty} u_j\not\equiv\infty$.*

PROOF. Denote the limit function by u. First we see from Lebesgue's monotone convergence theorem that u has mean-value property in D. Further, if $u(a)<\infty$ for some $a\in D$, then Harnack's inequality implies that

$$\begin{aligned}\left(\frac{R}{R+r}\right)^{n-2}\frac{R-r}{R+r}\,\{u(a)-u_j(a)\}\ &\leqq\ u(x)-u_j(x)\\ &\leqq\ \left(\frac{R}{R-r}\right)^{n-2}\frac{R+r}{R-r}\,\{u(a)-u_j(a)\}\end{aligned}$$

on $B(a,R)\subsetneqq D$. Hence $\{u_j\}$ converges to u uniformly on such ball. Since D is connected, this must be true for all balls in D, and it follows that u is harmonic in D and $\{u_j\}$ converges to u locally uniformly in D.

If there is no a for which $u(a) < \infty$, then $u \equiv \infty$.

THEOREM 1.8. *Let $\{u_j\}$ be a sequence of harmonic functions on a domain D. If $\{u_j\}$ is uniformly bounded in D, that is,*

$$M = \sup_j \|u_j\|_\infty < \infty,$$

then there exists a subsequence $\{u_{j(k)}\}$ which converges locally uniformly in D.

PROOF. Denote the gradient of v by ∇v. Let $B(a, 2R) \subsetneqq D$. By mean-value property, we have for $x \in B(a, R) \subsetneqq D$

$$\begin{aligned} |\nabla u_j(x)| &= \frac{1}{|B(x,R)|} \left| \int_{B(x,R)} \nabla u_j(y)\, dy \right| \\ &= \frac{1}{|B(x,R)|} \left| \int_{S(x,R)} u_j \boldsymbol{n}\, dy \right| \\ &\leqq \frac{1}{|B(x,R)|} M |S(x,R)| = \frac{nM}{R}, \end{aligned}$$

where $\boldsymbol{n}$ denotes the normal on $S(x, R)$. Hence it follows that

$$|u_j(x) - u_j(y)| \leqq \left(\sup_{B(a,R)} |\nabla u_j| \right) |x - y| \leqq \frac{nM}{R} |x - y|$$

whenever $x, y \in B(a, R)$. This implies that $\{u_j\}$ is equiuniformly continuous on $B(a, R)$ and, in particular, equicontinuous on D. By appealing the Ascoli-Arzelà theorem, we can choose a subsequence $\{u_{j(k)}\}$ which converges locally uniformly in D.

3.2 Superharmonic functions

We say that a function s on G is superharmonic if

$(S.1)$ s is lower semicontinuous in G;

$(S.2)$ $-\infty < s \not\equiv \infty$ on any component of G;

$(S.3)$ s has the super-mean-value property in G, that is,

$$s(x) \geqq \frac{1}{|B(x,r)|} \int_{B(x,r)} s(y)\, dy \qquad \text{whenever } \overline{B(x,r)} \subsetneqq G.$$

We say that s is subharmonic in G if $-s$ is superharmonic in G. Then u is harmonic in G if and only if u is superharmonic and subharmonic in G.

LEMMA 2.1. *If s is superharmonic in G, then it is locally integrable in G.*

PROOF. By considering each component of G, at first, we may assume that G is connected. Consider the set

$$E = \left\{ x \in G : \int_B |s(y)|\, dy < \infty \quad \text{for some ball } B = B(x,r) \text{ with } \overline{B} \subsetneqq G \right\}.$$

Since s is lower semicontinuous and $s > -\infty$ on G, s is locally bounded below on G. Since $(S.2)$, together with $(S.3)$, assures the existence of $x \in G$ for which

$$\infty > s(x) \geqq \frac{1}{|B(x,r)|} \int_{B(x,r)} s(y)\, dy$$

whenever $\overline{B(x,r)} \subsetneqq G$, so that

$$\int_{B(x,r)} |s(y)| dy < \infty$$

for such ball $B(x,r)$. Thus E is not empty. It is not so difficult to see that E is open and closed in the relative topology of G. Since G is connected by assumption, $E = G$, so that s is locally integrable in G.

THEOREM 2.1. *Let s be a function on G satisfying $(S.1)$ and $(S.2)$. Then the following are equivalent.*

(1) *s is superharmonic in G.*

(2) $s(x) \geqq \dfrac{1}{|S(x,r)|} \displaystyle\int_{S(x,r)} s(y)\, dS(y)$ *whenever* $\overline{B(x,r)} \subsetneqq G$.

(3) $u \in L^1_{loc}(G)$, $s(x) = \lim\limits_{r \to 0} \dfrac{1}{|B(x,r)|} \displaystyle\int_{B(x,r)} s(y)\, dy$ *for every* $x \in G$ *and*

$$\int s(x) \Delta\varphi(x)\, dx \leqq 0 \qquad \text{for any nonnegative function } \varphi \in C_0^\infty(G).$$

PROOF. First suppose $s \in C^2(G)$. If s is superharmonic in G, then we show that $\Delta s \leqq 0$ in G, so that (1) implies (3) in this case. In fact, if $x \in G$, then by Taylor's formula, we see that

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} s(y)\, dy = s(x) + \frac{\Delta s(x)}{2(n+2)} r^2 + o(r^2)$$

for small r, so that $(S.3)$ yields

$$\Delta s(x) \leqq 0.$$

Conversely, if $\Delta s \leqq 0$, then we apply Green's formula with s and $N_x - c$ in the ring domain $R(a,b)$ to obtain

$$0 \geqq \int_{R(a,b)} \{(\Delta s)(N_x - c) - s(\Delta N_x)\}\, dy = \int_{\partial R(a,b)} \{s_{\mathbf{n}}(N_x - c) - s(N_x)_{\mathbf{n}}\}\, dS(y),$$

where c is a constant so chosen that $N_x = c$ on $S(x,b)$. Here note that

$$\int_{\partial R(a,b)} s_{\mathbf{n}}(N_x - c)\, dS(y) = -(U_2(a) - c)\int_{B(x,a)} \Delta s\, dy \geqq 0,$$

so that we obtain

$$\frac{1}{|S(x,b)|}\int_{S(x,b)} s(y)\, dS(y) \leqq \frac{1}{|S(x,a)|}\int_{S(x,a)} s(y)\, dS(y).$$

By letting a tend to zero, we establish

$$\frac{1}{|S(x,b)|}\int_{S(x,b)} s(y)\, dS(y) \leqq s(x).$$

Consequently (1), (2) and (3) is equivalent for $s \in C^2(G)$.

Now we conquer the general case, and suppose (1) holds. By (2.1),

$$s(x) \geqq \frac{1}{|B(x,r)|}\int_{B(x,r)} s(y)\, dy \geqq \inf_{B(x,r)} s,$$

so that (3) holds by the lower semicontinuity of s. Since $s \in L^1_{loc}(G)$ by Lemma 2.1, as in the proof of Theorem 1.1, we may consider the convolution $s * \psi_\delta$; here we may assume that $\psi' \leqq 0$. Since (1) holds, we have for $\overline{B(x,r)} \subseteqq G_\delta$,

$$\begin{aligned}
\int_{B(x,r)} (s*\psi_\delta)(y)\, dy &= \int \left(\int_{B(x,r)} s(y-z)\, dy\right)\psi_\delta(z)\, dz \\
&\leqq \int |B(x,r)|\, s(x-z)\psi_\delta(z)\, dz \\
&= |B(x,r)|(s*\psi_\delta)(x),
\end{aligned}$$

which implies that $s * \psi_\delta$ is superharmonic in G_δ. Thus

$$\int (s*\psi_\delta)\Delta\varphi\, dx \leqq 0$$

for any $\varphi \in C_0^\infty(G_\delta)$ such that $\varphi \geqq 0$. Letting $\delta \to 0$ and noting that $s * \psi_\delta \to s$ in $L^1_{loc}(G)$, we have

$$\int s\Delta\varphi\, dx \leqq 0,$$

and hence (3) is fulfilled. Now the implication (1) $\Rightarrow$ (3) is proved.

Conversely, if (3) holds, then

$$\Delta(s*\psi_\delta) = s*(\Delta\psi_\delta) = \int s(y)\Delta\psi_\delta(x-y)\, dy \leqq 0$$

on G_δ, so that $s * \psi_\delta$ is superharmonic in G_δ by the above considerations given for C^2 functions. Hence

$$(s*\psi_\delta)(x) \geqq \frac{1}{|B(x,r)|}\int_{B(x,r)} (s*\psi_\delta)(y)\, dy$$

whenever $\overline{B(x,r)} \subseteqq G_\delta$. On the other hand,

$$\begin{aligned} s * \psi_\delta(x) &= \int s(x-y)\psi_\delta(y)\,dy \\ &= -\int \left(\int_{B(0,r)} s(x-y)\,dy \right) \psi'_\delta(r)\,dr \\ &\leqq \sup_{0<r<\delta} \frac{1}{|B(x,r)|} \int_{B(x,r)} s(y)\,dy. \end{aligned}$$

Thus we have

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} s * \psi_\delta\,dy \leqq s * \psi_\delta(x) \leqq \sup_{0<r<\delta} \frac{1}{|B(x,r)|} \int_{B(x,r)} s(y)\,dy.$$

If follows from (3) and the fact that $s * \psi_\delta \to s$ in $L^1_{loc}(G)$ as $\delta \to 0$ that

$$\begin{aligned} s(x) &\geqq \limsup_{\delta \to 0}\,(s * \psi_\delta)(x) \geqq \liminf_{\delta \to 0}\,(s * \psi_\delta)(x) \\ &\geqq \liminf_{\delta \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} s * \psi_\delta\,dy \\ &\geqq \frac{1}{|B(x,r)|} \int_{B(x,r)} s(y)\,dy. \end{aligned}$$

This implies, with the aid of (3) again, that

$$s(x) = \lim_{\delta \to 0}\,(s * \psi_\delta)(x).$$

Thus Fatou's lemma gives

$$\begin{aligned} s(x) &= \lim_{\delta \to 0}\,(s * \psi_\delta)(x) \\ &\geqq \liminf_{\delta \to 0} \frac{1}{|S(x,r)|} \int_{S(x,r)} s * \psi_\delta\,dS(y) \\ &\geqq \frac{1}{|S(x,r)|} \int_{S(x,r)} s(y)\,dS(y). \end{aligned}$$

Thus (3) implies (2) and then (1).

REMARK 2.1. Note that

$$\begin{aligned} \Delta U_\alpha(x) &= U''_\alpha(|x|) + \frac{n-1}{r} U'_\alpha(|x|) \\ &= (\alpha - n)(\alpha - 2)|x|^{\alpha-n-2} \end{aligned}$$

for $x \neq 0$. Hence, the α-kernel $U_\alpha(x)$ is superharmonic in $\mathbf{R}^n$ if $2 \leqq \alpha \leqq n$. Moreover, U_α is subharmonic in $\mathbf{R}^n - \{0\}$ if $0 < \alpha \leqq 2$.

COROLLARY 2.1. *Suppose* $U_\alpha\mu \not\equiv \infty$, $\mu \in \mathcal{M}$. *Then* $U_\alpha\mu$ *is superharmonic in* $\mathbf{R}^n$ *in case* $2 \leqq \alpha \leqq n$, *and* $U_\alpha\mu$ *is subharmonic outside the support of* μ *in case* $0 < \alpha \leqq 2$.

REMARK 2.2. We compute the potential

$$u(x) = \frac{1}{|\mathbf{S}|}\int_{\mathbf{S}} N(x-y)dS(y).$$

In view of Corollary 2.1, u is harmonic outside $\mathbf{S}$. Since u can be considered as a function of $r = |x|$, we see by polar coordinates that

$$(r^{n-1}u'(r))' = 0.$$

Hence $r^{n-1}u'(r)$ is constant, so that u is of the form $c_1N(r) + c_2$ for some constants c_1 and c_2. If we note that u is continuous at the origin and $u(x) - N(x)$ vanishes at infinity, then it follows that

$$u(x) = \min\{N(1), N(|x|)\}. \tag{2.1}$$

REMARK 2.3. If u is harmonic in G, then $|u|^p$ is subharmonic in G when $1 \leqq p < \infty$; in fact, by the mean-value property and Hölder's inequality, we have for any $B = B(x,r)$ with closure in G,

$$\begin{aligned} |u(x)|^p &= \left(\frac{1}{|B|}\int_B |u(y)|\,dy\right)^p \\ &\leqq \frac{1}{|B|^p}\left(\int_B dy\right)^{p-1}\int_B |u(y)|^p\,dy \\ &= \frac{1}{|B|}\int_B |u(y)|^p\,dy. \end{aligned}$$

In general, if φ is increasing and convex on the real line and s is subharmonic in G, then $\varphi(s)$ is subharmonic in G, because Jensen's inequality yields

$$\varphi(s(x)) \leqq \varphi\left(\frac{1}{|B|}\int_B s(y)\,dy\right) \leqq \frac{1}{|B|}\int_B \varphi(s(y))\,dy$$

for any ball $B = B(x,r)$ with closure in G.

REMARK 2.4. If $f(z)$ is holomorphic in the plane region G, then $\log|f|$ is subharmonic in G. Hence, noting that $\varphi(t) = e^{pt}$ is increasing and convex for $p > 0$, we see that $|f|^p = \exp(p\log|f|)$ is subharmonic in G for all $p > 0$.

THEOREM 2.2 (minimum principle). *Let s be superharmonic in a bounded domain D, and set* $\overline{s}(y) = \liminf_{x\to y, x\in D} s(x)$ *for* $y \in \partial D$. *Then*

$$\inf_D s = \inf_{\partial D} \overline{s}; \tag{2.2}$$

if s attains minimum inside D, then s is constant.

PROOF. Denote by M the left-hand side of (2.2) and by N the right-hand side of (2.2). Then $M \leqq N$, clearly. Suppose $M < N$. By the lower semicontinuity of s, there exists a point $x_0 \in D$ such that $M = s(x_0)$, from which it is seen that M is finite. Now consider the set

$$E = \{x \in D : s(x) = M\}.$$

Since $x_0 \in E$, E is not empty. Further it is easy to see that E is (relatively) closed in D. In view of the super-mean-value property, we have for $x \in E$,

$$\int_{B(x,r)} [s(y) - s(x)]\ dy \leqq 0$$

whenever $\overline{B(x,r)} \subseteqq D$. Since $s(y) \geqq M = s(x)$ on D,

$$s = M \tag{2.3}$$

holds for almost every $y \in B(x,r)$. Hence it follows from (3) of Theorem 2.1 that (2.3) holds everywhere on $B(x,r)$. Therefore E is open, and then $E = D$ by the connectedness of D. This means that s is constant and hence $N = M$, from which a contradiction follows. The second assertion can be proved similarly.

If an open set G is not bounded, then we define $\partial^* G = \partial G \cup \{\infty\}$ and $\overline{u}(\infty) = \liminf_{|x|\to\infty, x\in G} u(x)$ for a function u on G; if G is bounded, then we set $\partial^* G = \partial G$.

COROLLARY 2.2. *Let s be superharmonic in an open set G. If $\overline{s} \geqq 0$ on $\partial^* G$, then $s \geqq 0$ on G.*

COROLLARY 2.3. *Let G be a bounded domain in $\mathbf{R}^n$, and let u be a continuous function on $\overline{G}$. If u is harmonic in G and attains maximum or minimum inside G, then it is constant.*

Next we are concerned with the representation for superharmonic functions. First we show the following result, which is an easy consequence of (1.1).

LEMMA 2.2. *Let μ be a measure on $\mathbf{R}^n$ such that $|U_2\mu| \not\equiv \infty$. Then*

$$\Delta U_2 \mu = -a_n \omega_n \mu$$

in the sense of distributions.

If s is superharmonic in G, then $\mu = -(a_n\omega_n)^{-1}\Delta s$ is called the Riesz measure of s.

THEOREM 2.3 (Riesz decomposition theorem). *Let s be a superharmonic function on G, and denote the Riesz measure of s by μ. Then for each relatively compact open* *subset D of G, there exists a harmonic function h_D on D such that*

$$s = \int_D N(x-y)\, d\mu(y) + h_D \qquad \text{on } D.$$

With the aid of Theorems 8.1 and 8.2, we have the following results.

COROLLARY 2.4. *If s is a superharmonic function on G, then, for any $x \in G$, there exists a set $E = E_x \subseteqq \mathbf{S}$ such that $C_2(E) = 0$ and*

$$\lim_{r\to 0} s(x + r\xi) = s(x) \qquad \text{whenever } \xi \in \mathbf{S} - E.$$

COROLLARY 2.5. *Let μ be the Riesz measure of a superharmonic function s on G. Then, for any $x \in G$, there exists a set $E = E_x \subseteqq \mathbf{S}$ such that $C_2(E) = 0$ and*

$$\lim_{r\to 0} [N(r)]^{-1} s(x + r\xi) = \mu(\{x\}) \qquad \text{whenever } \xi \in \mathbf{S} - E.$$

LEMMA 2.4. *If s is superharmonic in $B(a, R)$, then*

$$s(x) \geqq \int_{S(a,r)} P_{B(a,r)}(x, y) s(y)\, dS(y)$$

whenever $x \in B(a, r)$, $0 < r < R$.

PROOF. Since s is lower semicontinuous on $S(a, r)$, by Lemma 1.1 in Chapter 2, there exists a sequence $\{f_j\}$ of continuous functions on $S(a, r)$ which increases to s. Define

$$u_j(x) = \int_{S(a,r)} P_{B(a,r)}(x, y) f_j(y)\, dS(y).$$

Then, in view of Theorem 1.3, u_j is harmonic in $B(a, r)$ and continuous on $\overline{B(a, r)}$. Note that

$$\liminf_{x\to y, x\in B(x,r)} s(y) \geqq s(y) \geqq f_j(y) = \lim_{x\to y, x\in B(x,r)} u_j(x)$$

for $y \in S(a, r)$, so that minimum principle implies that

$$s \geqq u_j \qquad \text{on } B(a, r).$$

By letting $j \to \infty$ and applying Lebesgue's monotone convergence theorem, we obtain the required result.

COROLLARY 2.6. *Let s be lower semicontinuous on G. Then s is superharmonic in G if and only if for any harmonic function h on a bounded open set G' with closure in G, $s \geqq h$ on $\partial G'$ implies $s \geqq h$ on G'.*

Proof. The only if part is an easy consequence of minimum principle of Theorem 2.2. To show the if part, let $B(a,r)$ be a ball with closure in G. In the proof of Lemma 2.4, $s \geqq u_j$ on $S(a,r)$, so that $s \geqq u_j$ on $B(a,r)$ by the assumption. Thus it follows that

$$s(x) \geqq \int_{S(a,r)} P_{B(a,r)}(x,y)s(y)\ dS(y)$$

for any $x \in B(a,r)$. In particular, if $x = a$, then

$$s(x) \geqq \frac{1}{|S(a,r)|} \int_{S(a,r)} s(y)\ dS(y).$$

Thus s has the super-mean-value property in G, and hence it is superharmonic in G.

The function $G_x(y)$ having appeared in the proof of Theorem 1.2 is called Green's function for $B = B(a,R)$, which is sometimes written as

$$G_x(y) = G(x,y) = G_B(x,y)$$

and has the following properties :

(G1) For $x \in B$, $G(x,\cdot)$ is harmonic in $B - \{x\}$ and superharmonic in B.

(G2) $G(x,y)$ is symmetric, that is, $G(x,y) = G(y,x)$ on $B \times B$.

(G3) For $x \in B$, $G(x,\cdot)$ vanishes on ∂B.

(G4) $G(x,y) \leqq M\dfrac{(R^2 - |x-a|^2)(R^2 - |y-a|^2)}{|x-y|^n}$.

Lemma 2.5. *Let μ be a measure on $B = B(a,R)$ such that*

$$\int_B (R^2 - |y-a|^2)\ d\mu(y) < \infty. \tag{2.4}$$

Then the Green potential

$$G\mu(x) = \int_B G(x,y)\ d\mu(y)$$

is superharmonic in B.

Theorem 2.3. *Let s be a superharmonic function on $B = B(a,R)$. If $s \geqq 0$ on B, then there exist $\mu \in \mathcal{M}(B)$ and $\nu \in \mathcal{M}(\partial B)$ such that*

$$s(x) = \int_B G(x,y)\ d\mu(y) + \int_{\partial B} P(x,y)\ d\nu(y)$$

whenever $x \in B$.

PROOF. Let μ denote the Riesz measure of s on B. For $0 < r < R$, consider

$$u_r(x) = s(x) - \int_{B(a,r)} G(x, y)\ d\mu(y). \tag{2.5}$$

Then, in view of Lemmas 2.3 and 2.5, u_r is harmonic in $B(a, r)$ and superharmonic in B. Since $\liminf_{x \to z, x \in B} u_r(x) \geqq 0$, minimum principle implies that $u_r \geqq 0$ on B. In particular,

$$s(0) \geqq \int_{B(a,r)} \{N(y) - N(R)\}\ d\mu(y),$$

so that

$$\int_{B(a,r)} (R - |y - a|)\ d\mu(y) < \infty.$$

It follows from Lemma 2.5 that the Green potential $G\mu(x)$ is superharmonic in B and

$$G\mu_{B(a,r)}(x) \uparrow G\mu(x) \leqq s \qquad \text{on } B.$$

Since u_r decreases as $r \uparrow R$, the limit function $u = \lim_{r \to R} u_r$ satisfies the mean-value property on B, so that u must be harmonic in B. In view of (2.5), we see that

$$s(x) = u(x) + \int_B G(x, y)\ d\mu(y).$$

Since $u \geqq 0$ on B, in view of Theorem 1.5, it is of the form

$$u(x) = \int_{\partial B} P(x, y)\ d\nu(y)$$

for some measure ν on ∂B. Thus the theorem is established.

3.3 Boundary limits of superharmonic functions

For $\xi \in \mathbf{S}$, define the truncated cone $\Gamma(\xi, \theta)$ of vertex at ξ with half-angle θ, $0 < \theta < \pi/2$, by

$$\Gamma(\xi, \theta) = \{x : (x - \xi, -\xi) \geqq |x - \xi| \cos\theta\} \cap B(\xi, 1).$$

We note here that if $\delta > 0$ is small enough, then there exists a positive constant $c = c(\delta)$ satisfying

$$1 - |x|^2 > c|x - \xi| \tag{3.1}$$

and

$$|x - y| > c|y - \xi| \tag{3.2}$$

whenever $x \in \Gamma(\xi, \theta) \cap B(\xi, \delta)$ and $y \in \mathbf{S} \cap B(\xi, \delta)$.

LEMMA 3.1. *h be a nonnegative nondecreasing function on $[0,\infty)$ such that*

$$h(r)r^{1-\beta} \leqq M \tag{3.3}$$

and

$$\int_r^1 h(t)t^{-\beta-1}\, dt \leqq Mh(r)r^{-\beta} \qquad \text{whenever } 0<r<1 \tag{3.4}$$

for some $\beta > 0$, where M is a positive constant. If μ is a measure on $\mathbf{R}^n$ satisfying

$$\lim_{r\to 0}\, [h(r)]^{-1}\mu(B(\xi,r)) = 0, \tag{3.5}$$

then

$$\lim_{r\to 0} \int_{B(\xi,1)} \frac{r}{(r+|\xi-y|)^\beta}\, d\mu(y) = 0.$$

PROOF. Set

$$\varepsilon(r) = \sup_{0<t<r}\, [h(t)]^{-1}\mu(B(\xi,t)) = 0.$$

Then it follows from (3.5) that

$$\lim_{r\to 0}\, \varepsilon(r) = 0.$$

Note here that

$$\begin{aligned}\limsup_{r\to 0} \int_{B(\xi,r)} \frac{r}{(r+|\xi-y|)^\beta}\, d\mu(y) &\leqq \limsup_{r\to 0}\, r^{1-\beta}\mu(B(\xi,r)) \\ &\leqq \limsup_{r\to 0}\, \varepsilon(r)h(r)r^{1-\beta} = 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\limsup_{r\to 0} \int_{B(\xi,1)-B(\xi,r)} \frac{r}{(r+|\xi-y|)^\beta}\, d\mu(y) &\leqq \limsup_{r\to 0}\, r\int_r^1 t^{-\beta}\, d\mu(B(\xi,t)) \\ &= \limsup_{r\to 0}\, r\int_r^1 \mu(B(\xi,t))d(-t^{-\beta}) \\ &= \limsup_{r\to 0}\, r\int_r^\delta \mu(B(\xi,t))d(-t^{-\beta}) \\ &= \limsup_{r\to 0}\, r\varepsilon(\delta)\int_r^\delta h(t)d(-t^{-\beta}) \leqq M\varepsilon(\delta),\end{aligned}$$

from which it follows that

$$\lim_{r\to 0} \int_{B(\xi,1)-B(\xi,r)} \frac{r}{(r+|\xi-y|)^\beta}\, d\mu(y) = 0.$$

LEMMA 3.2. *Let*

$$u(x) = \int_{\mathbf{S}} P(x,y)\, d\mu(y)$$

for $\mu \in \mathcal{M}(\mathbf{S})$. *If* $\xi \in \mathbf{S}$ *and*

$$\lim_{r \to 0} \frac{\mu(B(\xi, r))}{|\mathbf{S} \cap B(\xi, r)|} = 0, \tag{3.6}$$

then

$$\lim_{x \to \xi, x \in \Gamma(\xi, \theta)} u(x) = 0$$

for any $\theta \in (0, \pi/2)$; *in this case,* u *is said to have nontangential limit zero at* ξ.

PROOF. For $\delta > 0$, write

$$\begin{aligned} u(x) &\leqq \int_{\mathbf{S} \cap B(\xi, \delta)} P(x, y)\, d\mu(y) + \int_{\mathbf{S} - B(\xi, \delta)} P(x, y)\, d\mu(y) \\ &= u_1(x) + u_2(x). \end{aligned}$$

First we note that

$$\lim_{x \to \xi, x \in \mathbf{B}} u_2(x) = 0.$$

In view of (3.1) and (3.2), for $x \in \Gamma(\xi, \theta) \cap B(\xi, \delta)$ and $y \in \mathbf{S} \cap B(\xi, \delta)$, we find

$$u_1(x) \leqq M \int_{\mathbf{B} \cap B(\xi, \delta)} \frac{|x - \xi|}{(|x - \xi| + |y - \xi|)^n}\, d\mu(y)$$

with a positive constant M, if δ is small enough. Now, applying Lemma 3.1 with $h(r) = |\mathbf{S} \cap B(\xi, r)|$, we have

$$\lim_{x \to \xi, x \in \Gamma(\xi, \theta)} u_1(x) = 0.$$

Thus the lemma is proved.

By considering the derivative of μ with respect to the surface measure $dS|_{\mathbf{S}}$, we can write

$$\mu = f\, dS|_{\mathbf{S}} + \sigma, \tag{3.7}$$

where σ is singular and f is defined by

$$f(y) = \lim_{r \to 0} \frac{\mu(B(\xi, r))}{|\mathbf{S} \cap B(\xi, r)|}, \tag{3.8}$$

which exists for almost every $\xi \in \mathbf{S}$. Moreover, note that

$$\lim_{r \to 0} \int_{\mathbf{S} \cap B(\xi, r)} |f(y) - f(\xi)| dS(y) = 0 \tag{3.9}$$

for almost every $\xi \in \mathbf{S}$; if this is true, then such ξ is called a Lebesgue point for f (see Corollary 10.1 in Chapter 1).

THEOREM 3.1. *Let $\mu \in \mathcal{M}(\mathbf{S})$ and write it as (3.7). If we set*

$$u(x) = \int_{\mathbf{S}} P(x,y)\, d\mu(y),$$

then

$$\lim_{x \to y, x \in \Gamma(\xi,\theta)} u(x) = f(\xi) \qquad \text{whenever } 0 < \theta < \pi/2,$$

for almost every $\xi \in \mathbf{S}$; that is, u has a nontangential limit at almost every boundary point.

PROOF. Letting dS denote the surface measure on the unit sphere, we note first (cf. Theorem 10.3 in Chapter 1) that

$$\lim_{r \to 0} \frac{\sigma(B(\xi,r))}{S(B(\xi,r))} = 0$$

for almost every $\xi \in \mathbf{S}$, so that Lemma 3.2 shows that $\int_{\mathbf{S}} P(x,y)\, d\sigma(y)$ has nontangential limit zero at almost every $\xi \in \mathbf{S}$. Since

$$\left| \int_{\mathbf{S}} P(x,y) f(y) dS(y) - f(\xi) \right| \leqq \int_{S(0,r)} P(x,y) |f(y) - f(\xi)| dS(y),$$

it follows from Lemma 3.2 that

$$\lim_{r \to 0} \int_{\mathbf{S} \cap B(\xi,r)} P(x,y)\, |f(y) - f(\xi)|\, dS(y) = 0$$

as long as (3.9) holds. Thus the theorem is proved.

Next we deal with fine boundary limits for Green's potentials in the unit ball $\mathbf{B}$.

THEOREM 3.2. *Let μ be a measure on $\mathbf{B}$ such that $G\mu(x) \not\equiv \infty$ on $\mathbf{B}$. Then for almost every $\xi \in \mathbf{S}$, there exists a set $E(\xi) \subseteqq \mathbf{B}$ which is 2-thin, or simply thin, at ξ such that*

$$\lim_{x \to \xi, x \in \Gamma(\xi,\theta) - E} G\mu(x) = 0 \qquad \text{whenever } 0 < \theta < \pi/2;$$

that is, $G\mu(x)$ has nontangential fine limit zero at almost every boundary point.

Before giving a proof, we prepare the following lemmas.

LEMMA 3.3. *Let $\mu \in \mathcal{M}(B)$ be as in Theorem 3.2. For $\delta > 0$, set*

$$A(\delta) = \left\{ \xi \in \mathbf{S} : \int_{\mathbf{B}} \frac{(1 - |y|^2)^{1+\delta}}{|\xi - y|^{n-1+\delta}}\, d\mu(y) = \infty \right\}.$$

Then $A(\delta)$ has $(n-1)$-dimensional measure zero.

In fact,

$$\begin{aligned}
&\int_{\mathbf{S}}\left(\int_{\mathbf{B}} \frac{(1-|y|^2)^{1+\delta}}{|\xi-y|^{n-1+\delta}}\, d\mu(y)\right) dS(\xi) \\
=\ & \int_{\mathbf{B}}(1-|y|^2)^{1+\delta}\left(\int_{\mathbf{S}} |\xi-y|^{-n+1-\delta}\, dS(\xi)\right) d\mu(y) \\
\leqq\ & \int_{\mathbf{B}}(1-|y|^2)^{1+\delta}\left(\int_{\mathbf{S}} M[(1-|y|^2)+|\xi-(y/|y|)|]^{-n+1-\delta}\, dS(\xi)\right) d\mu(y) \\
\leqq\ & M\int_{\mathbf{B}}(1-|y|^2)\, d\mu(y) < \infty.
\end{aligned}$$

LEMMA 3.4. *Let ν be a measure on $\mathbf{B}$ with finite total mass. If we set*

$$F = \left\{\xi \in \mathbf{S} : \limsup_{r\to 0}\ r^{1-n}\nu(\mathbf{B}\cap B(\xi,r)) > 0\right\},$$

then E has $(n-1)$-dimensional measure zero.

PROOF. Let $0<\delta<1$ and consider

$$F(\delta) = \left\{\xi \in \mathbf{S} : \limsup_{r\to 0}\ r^{1-n}\nu(\mathbf{B}\cap B(\xi,r)) > \delta\right\}.$$

We have only to show that $|F(\delta)| = 0$. For any $\varepsilon > 0$ and $\xi \in F(\delta)$, there exists $r = r(\xi)$ such that $0 < r < \varepsilon$ and

$$r^{1-n}\nu(B\cap B(\xi,r)) > \delta.$$

By a covering lemma (see Theorem 10.1 in Chapter 1), we can choose a disjoint subfamily $\{B(\xi_j, r_j)\}$ for which

$$\bigcup_j B(\xi, 5r_j) \supseteqq F(\delta).$$

Note here that

$$\begin{aligned}
\nu(\bigcup_j B\cap B(\xi,r_j)) &= \sum_j \nu(B\cap B(\xi,r_j)) \\
&\geqq \delta\sum_j r_j^{n-1} \\
&\geqq M\delta\sum_j |\mathbf{S}\cap B(\xi_j,5r_j)| \\
&\geqq M\delta|F(\delta)|.
\end{aligned}$$

On the other hand, since $B(\xi,r_j) \subseteqq \mathbf{B} - B(0,1-\varepsilon)$,

$$\nu(\bigcup_j \mathbf{B}\cap B(\xi,r_j)) \leqq \nu(\mathbf{B}-B(0,1-\varepsilon)) \to 0$$

as $\varepsilon \to 0$, so that it follows that

$$|F(\delta)| = 0$$

as required.

PROOF OF THEOREM 3.2. Let $\xi \in \mathbf{S} - (A(\delta) \cup F)$. For $\theta \in (0, \pi/2)$, take θ' such that $\theta < \theta' < \pi/2$. Then find $c > 0$ for which

$$B(x, c(1 - |x|)) \subseteqq \Gamma(\xi, \theta') \qquad \text{whenever } x \in \mathbf{B} \cap \Gamma(\xi, \theta).$$

Write

$$\begin{aligned} G\mu(x) &= \int_{B(x,c(1-|x|))} G(x,y)\ d\mu(y) + \int_{\mathbf{B}-B(x,c(1-|x|))} G(x,y)\ d\mu(y) \\ &= g_1(x) + g_2(x). \end{aligned}$$

Note that

$$g_2(x) \leqq M \int_{\mathbf{B}-B(x,c(1-|x|))} \frac{(1-|x|^2)(1-|y|^2)}{[|\xi - y| + (1-|x|)]^n}\ d\mu(y).$$

Since $\xi \notin F$, Lemma 3.1 proves, as in the proof of Theorem 3.1,

$$\lim_{x \to \xi, x \in \Gamma(\xi,\theta)} g_2(x) = 0.$$

On the other hand,

$$g_1(x) \leqq M \int_{B(x,c(1-|x|))} N(x - y)\ d\mu(y).$$

Since $\xi \notin A(\delta)$, we see that

$$\int_{\Gamma(\xi,\theta')\cap B(\xi,\varepsilon)} N(\xi - y)\ d\mu(y) < \infty.$$

In view of fine limit results (see Theorems 5.1 and 6.2), we can find a set $E(\theta) \subseteqq \Gamma(\xi, \theta)$ which is thin at ξ such that

$$\lim_{x \to \xi, x \in \Gamma(\xi,\theta)-E(\theta)} g_1(x) = 0.$$

Letting $\{\theta_j\}$ be a sequence which converges to $\pi/2$, we can find $\{r_j\}$ such that

$$E = \bigcup_j E(\theta_j) \cap B(\xi, r_j)$$

is thin at ξ. Since E has all the required properties, the theorem is obtained.

COROLLARY 3.1. *Let μ be a measure on $\mathbf{B}$ such that $G\mu(x) \not\equiv \infty$ on $\mathbf{B}$. Then for almost every $\xi \in \mathbf{S}$, $G\mu(x)$ has limit zero along almost every ray terminating at ξ and passing a point of $\mathbf{B}$.*

COROLLARY 3.2. *Let s be a nonnegative superharmonic function on $\mathbf{B}$. Then, for almost every $\xi \in \mathbf{S}$, s has a finite limit along almost every ray terminating at ξ and passing a point of $\mathbf{B}$.*

THEOREM 3.3. *Let μ be a measure on $\mathbf{B}$ such that $G\mu(x) \not\equiv \infty$ on $\mathbf{B}$. Then*

$$\lim_{r\to 1} G\mu(r\xi) = 0$$

for almost every $\xi \in \mathbf{S}$.

PROOF. Consider the set

$$E = \left\{ \xi \in \mathbf{S} : \limsup_{r\to 1} \int_{B(r\xi,(1-r)/2)} N(r\xi - y)\, d\mu(y) > 0 \right\}.$$

In view of the proof of Theorem 3.2, we have only to show that E is of surface measure zero. This can be carried out in the same way as in the proof of Lemma 3.4. In fact, consider for $\delta > 0$,

$$E(\delta) = \left\{ \xi \in \mathbf{S} : \limsup_{r\to 1} \int_{B(r\xi,(1-r)/2)} N(r\xi - y)\, d\mu(y) > \delta \right\}.$$

For given $\varepsilon > 0$ and $\xi \in E(\delta)$, there exists $r = r(\xi)$ such that $1 - \varepsilon < r < 1$ and

$$\int_{B(r\xi,(1-r)/2)} N(r\xi - y)\, d\mu(y) > \delta.$$

Then we see that

$$\begin{aligned} \int_{B(r\xi,(1-r)/2)} N(r\xi - y)\, d\mu(y) &= \int_0^{(1-r)/2} N(t)\, d\mu(B(r\xi, t)) \\ &\leqq M \int_0^{(1-r)/2} \mu(B(r\xi, t)) t^{1-n}\, dt. \end{aligned}$$

Hence there exists $t = t(\xi)$ such that $0 < t < (1-r)/2$ and

$$\mu(B(r\xi, t)) \geqq M^{-1}\delta[(1-r)/2]^{-1} t^{n-1},$$

so that

$$\int_{B(r\xi,t)} (1 - |y|^2)\, d\mu(y) \geqq M\delta t^{n-1}.$$

Denote by $S^*(\xi)$ the radial projection of $B(r(\xi)\xi, t(\xi))$. Then $\{S^*(\xi)\}$ covers $E(\delta)$. Consequently, by a covering lemma (see Theorem 10.1 in Chapter 1), we can choose a disjoint subfamily $\{S^*(\xi_j)\}$ for which

$$\bigcup_j 5S^*(\xi_j) \supseteqq E(\delta),$$

where $5S^*(\xi_j)$ denotes the spherical cap centered at ξ_j and expanded 5 times $S^*(\xi_j)$. Note here that

$$\begin{aligned} \sum_j t(\xi_j)^{n-1} &\leqq [M\delta]^{-1} \sum_j \int_{B(r(\xi_j)\xi_j, t(\xi_j))} (1-|y|^2)\, d\mu(y) \\ &\leqq [M\delta]^{-1} \sum_j \int_{B-B(0,1-2\varepsilon)} (1-|y|^2)\, d\mu(y), \end{aligned}$$

which tends to zero as $\varepsilon \to 0$. Hence we find

$$|E(\delta)| = 0$$

as required.

COROLLARY 3.3. *Let s be a nonnegative superharmonic function on $\mathbf{B}$. Then $s(r\xi)$ has a finite limit as $r \to 1$ for almost every $\xi \in \mathbf{S}$.*

THEOREM 3.4. *Let $n = 2$ and μ be a measure on the unit disc $\mathbf{B}$ such that $G\mu(x) \not\equiv \infty$ on $\mathbf{B}$. If γ is a curve in $\mathbf{B}$ tending to a boundary point ξ, then*

$$\liminf_{x \to \xi, x \in \gamma} (1-|x|) G\mu(x) = 0.$$

PROOF. For simplicity, let $d(x) = 1-|x|$ denote the distance of x from the boundary. As in the proof of Theorem 3.2, we write

$$\begin{aligned} G\mu(x) &= \int_{B(x,d(x)/2)} G(x,y)\, d\mu(y) + \int_{\mathbf{B}-B(x,d(x)/2)} G(x,y)\, d\mu(y) \\ &= g_1(x) + g_2(x). \end{aligned}$$

In view of (G4), we have

$$d(x)|g_2(x)| \leqq M \int_{\mathbf{B}-B(x,d(x)/2)} \frac{d(x)^2}{|x-y|^2}\, d(y) d\mu(y).$$

Since $d(x)/|x-y|$ is bounded on $\mathbf{B} - B(x, d(x)/2)$, we apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{x \to \mathbf{S}} d(x) g_2(x) = 0.$$

To treat g_1, we use the Green capacity C_G defined by

$$C_G(E) = \inf\ \nu(\mathbf{B})$$

for $E \subseteqq \mathbf{B}$, where the infimum is taken over all measures ν on $\mathbf{B}$ for which

$$\int_{\mathbf{B}} G(x,y)\, d\nu(y) \geqq 1 \qquad \text{whenever } x \in E.$$

Since $G\mu(x) \not\equiv \infty$,

$$\int_{\mathbf{B}} d(y)\ d\mu(y) < \infty.$$

Hence we can take a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_j a_j \int_{B_j} d(y)\ d\mu(y) < \infty,$$

where $B_j = \{x : 2^{-j-1} < d(x) < 2^{-j+2}\}$. Now we consider the sets

$$E_j = \{x : 2^{-j} \leqq d(x) < 2^{-j+1}, d(x)g_1(x) > a_j^{-1}\}.$$

If $x \in E_j$, then

$$a_j^{-1} < d(x)|g_1(x)| \leqq M2^{-j} \int_{B_j} G(x,y)\ d\mu(y).$$

Hence it follows from the definition of capacity that

$$C_G(E_j) \leqq Ma_j 2^{-j}\mu(B_j) \leqq Ma_j \int_{B_j} d(y)\ d\mu(y),$$

so that

$$\sum_j C_G(E_j) < \infty.$$

Moreover, setting $E = \bigcup_j E_j$, we see that

$$\lim_{x\to\mathbf{S}, x\in\mathbf{B}-E} d(x)g_2(x) = 0.$$

Let L be the line $\{(t,0) : 0 < t < 1\}$ and $L_j = \{(t,0) : 2^{-j} < 1-t < 2^{-j+1}\}$. If $x \in L_j$, then

$$\int_{L_j} G(x,(r,0))\ dr \leqq M \int_{L_j} \log \frac{2d(x)}{|t-r|}\ dr = Md(x),$$

and hence we have

$$C_G(L \cap B_j) \geqq [Md(x)]^{-1}|L_j| = M > 0.$$

Now we see that

$$C_G(\gamma \cap B_j) \geqq C_G(L_j) \geqq M.$$

This implies that $\gamma \cap E_j$ is not empty for large j, and thus it follows that

$$\liminf_{x\to\xi, x\in\gamma} d(x)g_2(x) = 0.$$

3.4 Kelvin transform

In section 3.1, we defined the inversion with respect to the spherical surface $S(a, R)$ by

$$x^* = a + \frac{R^2}{|x-a|^2}(x-a).$$

Note here that

$$(4.1) \qquad |x^* - y^*| = \frac{|x-y|}{|x-a||y-a|}.$$

For a function u on an open set $G \subsetneqq \mathbf{R}^n - \{a\}$, define the Kelvin transform of u with respect to $S(a, R)$ by

$$u^*(x^*) = \left(\frac{R}{|x^*-a|}\right)^{n-2} u(x).$$

Further set $G^* = \{x^* : x \in G\}$. If $u \in C^2(G)$, then we see that

$$\Delta_{x^*} u^*(x^*) = \left(\frac{R}{|x^*-a|}\right)^{n+2} \Delta u(x).$$

THEOREM 4.1. *Let $G \subsetneqq \mathbf{R}^n - \{a\}$. If u is harmonic in G, then u^* is harmonic in G^*.*

COROLLARY 4.1. *If s is superharmonic in G, then s^* is superharmonic in G^*.*

This is an easy consequence of Theorem 4.1 and Corollary 2.6. In fact, suppose $s^* \geqq u^*$ on ∂U^* for a harmonic function u^* on a bounded open set U^* with closure in G^*. In view of Theorem 4.1, the Kelvin transform u is harmonic in U. Since $s \geqq u$ on ∂U, it follows from Theorem 2.2 that $s \geqq u$ on U, or $s^* \geqq u^*$ on U^*. Hence Corollary 2.6 implies that s^* is superharmonic in G^*.

Let $a = (-1, 0, ..., 0)$ and $b = a/2$. Consider the inversion with respect to $S(a, 1)$. If $x^* \in B(b, 1/2)$, then

$$\frac{1}{4} - |b - x^*|^2 = \frac{x_1}{|x-a|^2},$$

so that $B(b, 1/2)^*$ is the upper half space $\mathbf{H} = \{y = (y_1, ..., y_n) : y_1 > 0\}$. Recall that Poison's kernel for $B = B(b, 1/2)$ is given by

$$P_B(x^*, y^*) = \frac{2}{\omega_n} \frac{1/4 - |x^* - b|^2}{|x^* - y^*|^n}.$$

If u is a positive harmonic function on $\mathbf{H}$, then u^* is harmonic in B by Theorem 4.1 and is of the form

$$u^*(x^*) = \int_{S(b,1/2)} P_B(x^*, y^*) \, d\mu^*(y^*)$$

for some positive measure μ^* on $S(b,1/2)$. Hence

$$\begin{aligned} u(x) &= \frac{u^*(x^*)}{|x-a|^{n-2}} \\ &= \frac{2x_1}{\omega_n}\mu^*(\{a\}) + \frac{2x_1}{\omega_n}\int_{\partial\mathbf{H}} \frac{|y-a|^n}{|x-y|^n}\, d\mu^*(y^*) \\ &= \frac{2x_1}{\omega_n}\mu^*(\{a\}) + \frac{2x_1}{\omega_n}\int_{\partial\mathbf{H}} \frac{1}{|x-y|^n}\, d\mu(y), \end{aligned}$$

where

$$\mu(E) = \int_{E^*} |y^*-a|^{-n}\, d\mu^*(y^*)$$

for a Borel set $E \subseteq \mathbf{H}$. Here note that

$$\int_{\partial\mathbf{H}} |y-a|^{-n}\, d\mu(y) = \mu^*(S(b,1/2) - \{a\}) < \infty.$$

Thus we have the following representation of positive harmonic functions on $\mathbf{H}$.

THEOREM 4.2. *If u is a positive harmonic function on $\mathbf{H}$, then there exist a nonnegative number c and a measure μ on $\partial\mathbf{H}$ such that*

$$u(x) = cx_1 + \frac{2x_1}{\omega_n}\int_{\partial\mathbf{H}} \frac{1}{|x-y|^n}\, d\mu(y)$$

for $x \in \mathbf{H}$.

COROLLARY 4.2. *If $x \in \mathbf{H}$, then*

$$\frac{2x_1}{\omega_n}\int_{\partial\mathbf{H}} \frac{1}{|x-y|^n}\, dy = 1.$$

Green's function for $B = B(b,1/2)$ is given by

$$G(x,y) = |x-y|^{2-n} - \left(\frac{|\tilde{x}-b|}{1/2}\right)^{n-2} |\tilde{x}-y|^{2-n},$$

where $\tilde{x}$ denotes the inversion of x with respect to $S(b,1/2)$. Then note that

$$|x-a| = 2|x-b||\tilde{x}-a|$$

and

$$\begin{aligned} \tilde{x}^* &= a + \frac{\tilde{x}-a}{|\tilde{x}-a|^2} \\ &= a + \frac{[b+(x-b)/4|x-b|^2]-a}{|\tilde{x}-a|^2} \\ &= a + \frac{x-(4|x-b|^2+1)b}{|x-a|^2}. \end{aligned}$$

Thus we see that

$$\tilde{x}^* = \overline{x^*},$$

where $\overline{y}$ denotes the reflection of y with respect to the hyperplane $\partial\mathbf{H}$.

THEOREM 4.3. *If s is a positive superharmonic function on $\mathbf{H}$, then there exist a nonnegative number c, a measure μ on $\partial\mathbf{H}$ and a measure ν on $\mathbf{H}$ such that*

$$s(x) = cx_1 + \frac{2x_1}{\omega_n}\int_{\partial\mathbf{H}} \frac{1}{|x-y|^n}\, d\mu(y) + \int_{\mathbf{H}} G_{\mathbf{H}}(x,y)\, d\nu(y)$$

for $x \in \mathbf{H}$, where

$$G_{\mathbf{H}}(x,y) = \begin{cases} |x-y|^{2-n} - |\overline{x}-y|^{2-n} & n \geqq 3, \\ \log(|\overline{x}-y|/|x-y|) & n = 2, \end{cases}$$

$$\int_{\partial\mathbf{H}} |y-e|^{-n}\, d\mu(y) < \infty$$

and

$$\int_{\mathbf{H}} y_1|y-e|^{-n}\, d\nu(y) < \infty.$$

PROOF. In view of Corollary 4.1, s^* is superharmonic in $B = B(b,1/2)$, and hence by Theorem 2.3, s^* is of the form

$$s^*(x^*) = \int_{S(b,1/2)} P_B(x^*,y^*)\, d\mu^*(y^*) + \int_{B(b,1/2)} G_B(x^*,y^*)\, d\nu^*(y^*).$$

By Theorem 4.2, the first integral is of the form

$$|x-a|^{n-2}\left(cx_1 + \frac{2x_1}{\omega_n}\int_{\partial\mathbf{H}} \frac{1}{|x-y|^n}\, d\mu(y)\right).$$

Similarly,

$$G_B(x^*,y^*) = |x-a|^{n-2}|y-a|^{n-2}G_{\mathbf{H}}(x,y),$$

so that the second integral is of the form

$$|x-a|^{n-2}\left(\int_{\mathbf{H}} G_{\mathbf{H}}(x,y)\, d\nu(y)\right),$$

where

$$\int_{\mathbf{H}} y_1|y-a|^{-n}\, d\nu(y) = \int_B (1/4 - |y^*-b|^2)\, d\nu^*(y^*) < \infty.$$

REMARK 4.1. In Theorem 4.2, $P_{\mathbf{H}}(x,y) = \dfrac{2}{\omega_n}\dfrac{x_1}{|x-y|^n}$ is called the Poisson kernel for $\mathbf{H}$, and $G_{\mathbf{H}}(x,y)$ is called Green's function for $\mathbf{H}$.

3.5 Balayage

In this section, let $0 < \alpha \leqq 2$ and $\alpha < n$. It is important to note here that every potential is subharmonic outside its support.

THEOREM 5.1 (maximum principle). *Suppose $U_\alpha\mu < \infty$ μ-a.e. and s is positive superharmonic in $\mathbf{R}^n$. If $U_\alpha\mu \leqq s$ μ-a.e. on S_μ, then $U_\alpha\mu \leqq s$ everywhere on $\mathbf{R}^n$.*

PROOF. In view of Theorems 1.4 and 1.5 in Chapter 2, for given $\varepsilon > 0$, we can find a compact set $K \subseteqq S_\mu$ such that $\mu(S_\mu - K) < \varepsilon$, $U_\alpha(\mu|_K)$ is continuous and

$$U_\alpha\mu \leqq s \qquad \text{on } K.$$

Then $s - U_\alpha(\mu|_K)$ is superharmonic outside K, $\liminf_{|x|\to\infty} s - U_\alpha(\mu|_K) \geqq 0$ and

$$U_\alpha(\mu|_K) \leqq U_\alpha\mu \leqq s \qquad \text{on } K.$$

Thus minimum principle for superharmonic functions implies that

$$s - U_\alpha(\mu|_K) \geqq 0 \qquad \text{on } \mathbf{R}^n - K,$$

so that $U_\alpha(\mu|_K) \leqq s$ on $\mathbf{R}^n$. By considering a sequence $\{K_j\}$ of compact sets for which $K_j \uparrow S_\mu$, we obtain the required inequality.

COROLLARY 5.1. *Suppose $U_\alpha\mu < \infty$ μ-a.e. . If $U_\alpha\mu \leqq 1$ μ-a.e. on S_μ, then $U_\alpha\mu \leqq 1$ everywhere on $\mathbf{R}^n$.*

With the aid of Theorem 10.1 in Chapter 2 we obtain the following result.

THEOREM 5.2. *If K is a compact set in $\mathbf{R}^n$, then there exists a unique measure γ_K supported by K such that*

(i) $U_\alpha\gamma_K(x) = 1$ *α-q.e. on K;*

(ii) $U_\alpha\gamma_K(x) \leqq 1$ *on $\mathbf{R}^n$.*

The above measure γ_K is called the equilibrium measure of K; $U_\alpha\gamma_K$ is called the equilibrium potential of K.

Suppose $U_\alpha\mu < \infty$ μ-a.e. and

$$U_\alpha\mu \leqq U_\alpha\delta_a \qquad \mu\text{-a.e. on } S_\mu, \tag{5.1}$$

where $a \in \mathbf{R}^n - S_\mu$. By considering the inversion with respect to $S(a, 1)$, we see that

$$\begin{aligned} U_\alpha\mu(x) &= |x^* - a|^{n-\alpha} \int |x^* - y^*|^{\alpha-n}|y^* - a|^{n-\alpha}\, d\mu(y) \\ &= |x^* - a|^{n-\alpha} U_\alpha\mu^*(x^*) \end{aligned}$$

for a measure μ^* with compact support. By (5.1),

$$U_\alpha \mu^* \leqq 1 \qquad \mu^*\text{-a.e. on } S_{\mu^*}.$$

Thus maximum principle implies that the inequality holds everywhere on $\mathbf{R}^n$, so that

$$U_\alpha \mu \leqq U_\alpha \delta_a \qquad \text{on } \mathbf{R}^n.$$

Now, applying Theorem 10.3 in Chapter 2 with $f = U_\alpha \delta_a$, we have the following result.

THEOREM 5.3. *If a is not contained in a compact set K, then there exists a unique measure $\delta_{a,K} \in \mathcal{E}_\alpha(K)$ such that*

(i) $U_\alpha \delta_{a,K} = U_\alpha \delta_a$ *α-q.e. on K;*

(ii) $U_\alpha \delta_{a,K} \leqq U_\alpha \delta_a$ *on $\mathbf{R}^n$.*

We say that $\delta_{a,K}$ is the balayage of δ_a, or the sweeping of δ_a to K.

LEMMA 5.1. *For $\varphi \in C_0^\infty(\mathbf{R}^n)$, $\delta_{a,K}(\varphi)$ is the difference of lower semicontinuous functions on $\mathbf{R}^n - K$.*

In fact, for $x \in \mathbf{R}^n - K$,

$$\begin{aligned} U_\alpha \delta_{a,K}(x) &= \int U_\alpha \delta_{a,K}\, d\delta_x = \int U_\alpha \delta_x\, d\delta_{a,K} \\ &= \int U_\alpha \delta_{x,K}\, d\delta_{a,K} = \int U_\alpha \delta_{a,K}\, d\delta_{x,K} \\ &= \int U_\alpha \delta_a\, d\delta_{x,K} = U_\alpha \delta_{x,K}(a), \end{aligned}$$

so that $U_\alpha \delta_{a,K}(x)$ is continuous on $\mathbf{R}^n - K$. Hence we see that $\int U_\alpha \mu\, d\delta_{a,K} = \int U_\alpha \delta_{a,K}\, d\mu$ is lower semicontinuous on $\mathbf{R}^n - K$ for any $\mu \in \mathcal{E}_\alpha$. Since $\varphi \in \overline{\mathcal{E}}_\alpha$, $\delta_{a,K}(\varphi)$ is the difference of lower semicontinuous functions on $\mathbf{R}^n - K$.

LEMMA 5.2. *Let μ be a measure such that $S_\mu \cap K = \emptyset$ and $U_\alpha \mu \not\equiv \infty$, and set*

$$\mu_K(\varphi) = \int \delta_{a,K}(\varphi)\, d\mu(a)$$

for $\varphi \in C_0^\infty(\mathbf{R}^n)$. Then μ_K defines a measure on $\mathbf{R}^n$.

PROOF. Let B be a ball. If $\varphi \in C_0^\infty(\mathbf{R}^n)$, then we can find $M > 0$ such that

$$|\varphi| \leqq M U_\alpha \chi_B.$$

Hence we have

$$\begin{aligned} |\delta_{a,K}(\varphi)| &\leqq M \int U_\alpha \chi_B \, d\delta_{a,K} = M \int U_\alpha \delta_{a,K} \, d\chi_B \\ &\leqq M \int U_\alpha \delta_a \, d\chi_B = M U_\alpha \chi_B(a), \end{aligned}$$

so that

$$\int |\delta_{a,K}(\varphi)| \, d\mu(a) \leqq M \int_B U_\alpha \mu \, dx < \infty.$$

COROLLARY 5.2. *Let μ be a measure such that $S_\mu \cap K = \emptyset$ and $U_\alpha \mu \not\equiv \infty$. Then there exists a unique measure $\mu_K \in \mathcal{E}_\alpha(K)$ such that*

(i) $U_\alpha \mu_K = U_\alpha \mu$ *α-q.e. on K;*

(ii) $U_\alpha \mu_K \leqq U_\alpha \mu$ *on $\mathbf{R}^n$.*

In fact, the measure μ_K in Corollary 10.1 in Chapter 2 has the required properties, and the uniqueness is easy.

THEOREM 5.4 (domination principle). *Suppose $\mu \in \mathcal{E}_\alpha$ and $\nu \in \mathcal{M}$. If $U_\alpha \mu \leqq U_\alpha \nu$ μ-a.e. on S_μ, then $U_\alpha \mu \leqq U_\alpha \nu$ everywhere on $\mathbf{R}^n$.*

PROOF. As in the proof of Theorem 5.1, we may assume that $K = S_\mu$ is compact and $U_\alpha \mu$ is continuous. Then we have for $a \in \mathbf{R}^n - K$,

$$\begin{aligned} U_\alpha \mu(a) &= \int U_\alpha \delta_a \, d\mu = \int U_\alpha \delta_{a,K} \, d\mu \\ &= \int U_\alpha \mu \, d\delta_{a,K} \leqq \int U_\alpha \nu \, d\delta_{a,K} \\ &= \int U_\alpha \delta_{a,K} \, d\nu \leqq \int U_\alpha \delta_a \, d\nu = U_\alpha \nu(a), \end{aligned}$$

which was to be proved.

By Corollary 10.1 in Chapter 2, we have

COROLLARY 5.3. *Let F be a closed set in $\mathbf{R}^n$. If $\mu \in \mathcal{E}_\alpha$, then there exists a unique measure $\mu_F \in \mathcal{E}_\alpha(F)$ such that*

(i) $U_\alpha \mu_F = U_\alpha \mu$ *α-q.e. on F;*

(ii) $U_\alpha \mu_F \leqq U_\alpha \mu$ *on $\mathbf{R}^n$.*

THEOREM 5.5. *Let F be a closed set in $\mathbf{R}^n$. If $U_\alpha \mu \not\equiv \infty$, then there exists a unique measure μ_F supported by F such that*

(i) $U_\alpha \mu_F = U_\alpha \mu$ α-q.e. *on* F;

(ii) $U_\alpha \mu_F \leqq U_\alpha \mu$ *on* $\mathbf{R}^n$;

(iii) $\int U_\alpha \mu_F \, d\nu = \int U_\alpha \nu_F \, d\mu$ *for any* $\nu \in \mathcal{E}_\alpha$.

PROOF. Let $F_j = F \cap \overline{B(0,j)}$ and take a sequence $\{f_{j,k}\}$ of positive continuous functions on F_j which increases to $U_\alpha \mu$. By Theorem 10.3 in Chapter 2, we can find a measure $\mu_{j,k} \in \mathcal{E}_\alpha(F_j)$ such that

(5.2) $$U_\alpha \mu_{j,k} \geqq f_{j,k} \qquad \alpha\text{-q.e. on } F_j,$$

(5.3) $$U_\alpha \mu_{j,k} \leqq f_{j,k} \qquad \text{on } S_{\mu_{j,k}}.$$

Since $f_{j,k} \leqq U_\alpha \mu$ on F_j, we have by domination principle,

(5.4) $$U_\alpha \mu_{j,k} \leqq U_\alpha \mu \qquad \text{on } \mathbf{R}^n.$$

Moreover, we see that $\{U_\alpha \mu_{j,k}\}$ increases with k. Since $U_\alpha \chi_B \geqq A_B > 0$ on a ball B, we have

$$\begin{aligned} A_B \mu_{j,k}(B) &\leqq \int U_\alpha \chi_B \, d\mu_{j,k} \\ &= \int U_\alpha \mu_{j,k} \, d\chi_B \\ &\leqq \int_B U_\alpha \mu \, dy < \infty, \end{aligned}$$

so that we may assume that $\{\mu_{j,k}\}$ converges vaguely to a measure $\mu_j \in \mathcal{M}(F_j)$. Since $U_\alpha \mu_j = \liminf_{k \to \infty} U_\alpha \mu_{j,k}$ α-q.e., (5.2) and (5.4) imply that

$$U_\alpha \mu_j \geqq U_\alpha \mu \qquad \alpha\text{-q.e. on } F_j,$$

and

$$U_\alpha \mu_j \leqq U_\alpha \mu \qquad \text{on } \mathbf{R}^n.$$

By the choice of $\{f_{j,k}\}$, we can show that $\{U_\alpha \mu_j\}$ increases and

$$A_B \mu_j(B) \leqq \int_B U_\alpha \mu \, dy < \infty$$

for any ball B, so that $\{\mu_j\}$ converges vaguely to a measure μ_0 supported by F. As seen above, we have

$$U_\alpha \mu_0 \geqq U_\alpha \mu \qquad \alpha\text{-q.e. on } F,$$

and

$$U_\alpha \mu_0 \leqq U_\alpha \mu \qquad \text{on } \mathbf{R}^n.$$

Thus μ_0 satisfies (i) and (ii). To show that μ_0 satisfies (iii), we have only to note that

$$\begin{aligned}\int U_\alpha \mu_{j,k}\, d\nu &= \int U_\alpha \nu\, d\mu_{j,k} \\ &= \int U_\alpha \nu_F\, d\mu_{j,k} \\ &= \int U_\alpha \mu_{j,k}\, d\nu_F\end{aligned}$$

and let $k \to \infty$ and then $j \to \infty$.

We say that μ_F is the balayage of μ to F.

REMARK 5.1. Let F be a closed set in $\mathbf{R}^n$ and suppose $U_\alpha \mu \not\equiv \infty$. Then there exists a sequence $\{\mu_j\}$ in $\mathcal{E}_\alpha(F)$ for which each $U_\alpha \mu_j$ is continuous on $\mathbf{R}^n$ and $U_\alpha \mu_j$ increases to $U_\alpha \mu_F$.

REMARK 5.2. Suppose $U_\alpha \mu \not\equiv \infty$ and $U_\alpha \nu \not\equiv \infty$. Then

$$\int U_\alpha \mu_F\, d\nu = \int U_\alpha \nu_F\, d\mu. \tag{5.5}$$

3.6 The classical Dirichlet problem

If D is a domain of $\mathbf{R}^n$ and f is a function on ∂D, then the classical Dirichlet problem is that of finding a function h which is harmonic in D and

$$\lim_{x \to \xi, x \in D} h(x) = f(\xi) \tag{6.1}$$

for all $\xi \in \partial D$. If a solution exists, then it is unique.

THEOREM 6.1. *If the classical Dirichlet problem has a solution h, then the boundary function f is continuous.*

In fact, let $\xi_j \in \partial D$ tend to ξ and $f(\xi_j)$ tend to ℓ. For each j, we can find $x_j \in D$ such that $|x_j - \xi_j| < 1/j$ and $|h(x_j) - f(\xi_j)| < 1/j$. Then $x_j \to \xi$ and

$$f(\xi) = \lim_{j \to \infty} h(x_j) = \lim_{j \to \infty} f(\xi_j) = \ell,$$

so that f is continuous at ξ.

First we consider the Dirichlet problem in the case $n \geqq 3$ and $\alpha = 2$. Let D be a bounded domain in $\mathbf{R}^n$, and $F = \mathbf{R}^n - D$.

LEMMA 6.1. *If $a \in D$, then the balayage $\delta_{a,F}$ is supported by the boundary ∂D.*

In fact, $U_\alpha \delta_{a,F} = U_\alpha \delta_a$ is harmonic outside $\overline{D}$, so that

$$\delta_{a,F}(\mathbf{R}^n - \overline{D}) = 0.$$

THEOREM 6.2. *If f is continuous on F, then*

$$H_f(x) = \int f(y)\, d\delta_{x,F}(y) \tag{5.1}$$

is harmonic in D.

PROOF. If $f = U_2\mu$ with $\mu \in \mathcal{E}_2$, then Theorem 5.5 gives

$$\begin{aligned} H_f(x) &= \int U_2\mu \, d\delta_{x,F} = \int U_2\mu_F \, d\delta_{x,F} \\ &= \int U_2\delta_{x,F} \, d\mu_F = \int U_2\delta_x \, d\mu_F = U_2\mu_F(x). \end{aligned}$$

Since $S_{\mu_F} \subseteqq F$, $U_2\mu_F$ is harmonic in D, and so is H_f. For a general continuous function f on F, given $\varepsilon > 0$, take $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that

$$|f - \varphi| < \varepsilon \qquad \text{on } F.$$

Then $\varphi \in \overline{\mathcal{E}}_2$ and H_φ is harmonic in D as seen above. Moreover, since $\delta_{x,F}(\mathbf{R}^n) \leqq \delta_x(\mathbf{R}^n) = 1$ by Theorem 1.7 in Chapter 2, we have

$$|H_f - H_\varphi| \leqq \int |f - \varphi| \, d\delta_{x,F} < \varepsilon.$$

Thus it is seen that H_f is a uniform convergence limit of harmonic functions on D, so that H_f is harmonic in D.

We say that $x_0 \in \partial D$ is regular if $\delta_{x,F}$ converges vaguely to δ_{x_0} as $x \to x_0$, $x \in D$; we say that x_0 is irregular if x_0 is not regular.

COROLLARY 6.1. *Suppose every boundary point of D is regular. Then for a given continuous function f on F, H_f is the Dirichlet solution on D.*

THEOREM 6.3. *The following are equivalent for $x_0 \in F$.*

(i) *x_0 is regular;*

(ii) *For $y \in D$, $U_2\delta_{x_0,F}(y) = U_2\delta_{x_0}(y)$;*

(iii) *$\delta_{x_0,F} = \delta_{x_0}$.*

PROOF. First we show that (i) implies (ii). First note that (5.5) gives

$$s_y(x) \equiv U_2\delta_{x,F}(y) = \int U_2\delta_{x,F}\, d\delta_y = \int U_2\delta_{y,F}\, d\delta_x = U_2\delta_{y,F}(x) \tag{6.2}$$

for any x and y, which implies that $s_y(x)$ is superharmonic in $\mathbf{R}^n$. On the other hand, if $y \in D$ is fixed, then $|y-z|^{2-n}$ is continuous on F, so that

$$\begin{aligned} \lim_{x\to x_0, x\in D} s_y(x) &= \lim_{x\to x_0, x\in D} \int |y-z|^{2-n}\, d\delta_{x,F}(z) \\ &= \int |y-z|^{2-n}\, d\delta_{x_0}(z) \\ &= U_2\delta_y(x_0). \end{aligned}$$

Since $s_y(x) = U_2\delta_y(x)$ 2-q.e. on F, or, simply, q.e. on F, the super-mean-value property implies that

$$s_y(x_0) = U_2\delta_y(x_0),$$

so that (ii) is fulfilled.

To show the implication (ii) $\Rightarrow$ (iii), we note first that

$$U_2\delta_{x_0,F}(x) = U_2\delta_{x,F}(x_0) = U_2\delta_x(x_0) = U_2\delta_{x_0}(x)$$

holds q.e. on F. Hence the equality holds q.e. on $\mathbf{R}^n$ by (ii), so that $\delta_{x_0,F} = \delta_{x_0}$ and (iii) holds.

Finally we show that (iii) implies (i). Note by (6.2) that

$$\liminf_{x\to x_0} U_2\delta_{x,F}(z) \geqq U_2\delta_{x_0,F}(z) = U_2\delta_{x_0}(z)$$

for any z. Hence, if $\mu \in \mathcal{E}_2$ and $U_2\mu$ is continuous, then

$$\liminf_{x\to x_0} \int U_2\mu\, d\delta_{x,F} = \liminf_{x\to x_0} \int U_2\delta_{x,F}\, d\mu \geqq \int U_2\delta_{x_0}\, d\mu = U_2\mu(x_0)$$

and

$$\int U_2\mu\, d\delta_{x,F} = \int U_2\delta_{x,F}\, d\mu \leqq \int U_2\delta_x\, d\mu = U_2\mu(x),$$

so that

$$\lim_{x\to x_0} \int U_2\mu\, d\delta_{x,F} = U_2\mu(x_0).$$

Hence, for $\varphi \in C_0^\infty(\mathbf{R}^n)$, we have

$$\lim_{x\to x_0} \int \varphi\, d\delta_{x,F} = \int \varphi\, d\delta_{x_0},$$

which implies that $\delta_{x,F} \to \delta_{x_0}$ vaguely as $x \to x_0$.

COROLLARY 6.2. *Let E be the set of all points of ∂D which are irregular. Then $C_2(E) = 0$, that is, E is polar.*

In fact, for any $y \in D$, it suffices to note that

$$E \subseteqq \{x \in \partial D : U_2\delta_{x,F}(y) < U_2\delta_x(y)\}.$$

COROLLARY 6.3. *Suppose $U_2\mu \not\equiv \infty$. If x_0 is regular, then*

$$U_2\mu_F(x_0) = U_2\mu(x_0).$$

THEOREM 6.4. *Let $x_0 \in \partial D$. Then x_0 is irregular if and only if F is thin at x_0.*

PROOF. Suppose x_0 is irregular. Then for fixed $y \in D$,

$$U_2\delta_{x_0,F}(y) < U_2\delta_{x_0}(y).$$

If F is not thin at x_0, then we can find $F' \subseteqq F$ for which

$$\begin{aligned} U_2\delta_{x_0,F}(y) &= U_2\delta_{y,F}(x_0) \\ &= \lim_{x \to x_0, x \in F'} U_2\delta_{y,F}(x) \\ &= \lim_{x \to x_0, x \in F'} U_2\delta_{x,F}(y) \\ &= \lim_{x \to x_0, x \in F'} U_2\delta_x(y) \\ &= U_2\delta_{x_0}(y), \end{aligned}$$

which yields a contradiction. Thus F is thin at x_0.

Suppose F is thin at x_0. Then there exists a measure μ such that

$$\liminf_{x \to x_0, x \in F} U_2\mu(x) > U_2\mu(x_0).$$

By the lower semicontinuity, we can find $r > 0$ such that

$$U_2\mu(x) > \eta > U_2\mu(x_0) \qquad \text{for any } x \in K - \{x_0\},$$

where $K = F \cap \overline{B(x_0, r)}$. Consider the equilibrium measure γ_K in Theorem 5.2. Then

$$U_2\mu > \eta U_2\gamma_K \qquad \text{q.e. on } K,$$

so that maximum principle implies that

$$U_2\mu \geqq \eta U_2\gamma_K \qquad \text{on } \mathbf{R}^n.$$

In particular, $U_2\mu(x_0) \geqq \eta U_2\gamma_K(x_0)$. If x_0 is regular, then, since the balayage of γ_K to F is itself, Corollary 5.3 gives

$$U_2\gamma_K(x_0) = 1,$$

so that

$$U_2\mu(x_0) \geqq \eta.$$

Thus a contradiction follows.

COROLLARY 6.4. *Let $x_0 \in \partial D$. If F contains a cone with vertex at x_0, then x_0 is regular.*

COROLLARY 6.5. *If D is a ball B, then H_f is equal to the Poisson integral of f for B. Further,*

$$G(x, y) = U_2\delta_x(y) - U_2\delta_{x,F}(y)$$

is equal to Green's function for B, where $F = \mathbf{R}^n - B$.

COROLLARY 6.6. *If D has an irregular boundary point, then there is no solution even if f is continuous on F.*

3.7 Dirichlet problem in the plane

In view of Theorem 10.5 in Chapter 2, we have the following.

THEOREM 7.1. *Let K be a compact set in $\mathbf{R}^2$ with $C_2(K) > 0$. Then for $a \in \mathbf{R}^2 - K$, there exist a unique measure $\delta_{a,K} \in \mathcal{E}_{2,1}(K)$ and a number $\gamma_{a,K}$ such that*

(i) $U_2\delta_{a,K} = U_2\delta_a + \gamma_{a,K}$ *q.e. on K;*

(ii) $U_2\delta_{a,K} \leqq U_2\delta_a + \gamma_{a,K}$ *on $\mathbf{R}^2$.*

Note here that

$$\gamma_{a,K} \geqq \lim_{|x|\to\infty} \{U_2\delta_{a,K}(x) - U_2\delta_a(x)\} = 0.$$

For $a \in \mathbf{R}^2 - \overline{D}$, apply Theorem 7.1 with $K = \overline{D}$. Then $\delta_{a,K}$ is supported on ∂D, since (i) of Theorem 7.1 implies that $U_2\delta_{a,K}$ is harmonic in D. Hence it follows that $C_2(\partial D) > 0$. If $x \in D$, then

$$U_2\delta_{x,\partial D} = U_2\delta_x + \gamma_{x,\partial D} \qquad \text{q.e. on } \partial D.$$

Since $\lim_{|y|\to\infty} \{U_2\delta_{x,\partial D}(y) - U_2\delta_x(y)\} = 0$, we see that

$$\gamma_{x,\partial D} = 0.$$

Now define Green's function for D by

$$G(x, y) = U_2\delta_x(y) - U_2\delta_{x,\partial D}(y).$$

THEOREM 7.2. *For $x, y \in D$, $G(x,y) = G(y,x)$.*

PROOF. In view of (i) of Theorem 7.1,

$$\int U_2\delta_{x,\partial D}\ d\delta_{y,\partial D} = \int U_2\delta_x\ d\delta_{y,\partial D} = U_2\delta_{y,\partial D}(x)$$

and

$$\int U_2\delta_{y,\partial D}\ d\delta_{x,\partial D} = \int U_2\delta_y\ d\delta_{x,\partial D} = U_2\delta_{x,\partial D}(y).$$

Thus the required equality follows.

As before, we say that $x_0 \in \partial D$ is regular if $\delta_{x,\partial D} \to \delta_{x_0}$ vaguely as $x \to x_0$, $x \in D$.

THEOREM 7.3. *Let $x_0 \in \partial D$. Then x_0 is regular if and only if*

$$\lim_{x\to x_0, x\in D} G(x,y) = 0 \tag{7.1}$$

for any $y \in D$.

PROOF. If x_0 is regular, then

$$\lim_{x\to x_0, x\in D} G(x,y) = \lim_{x\to x_0, x\in D} G(y,x) = 0.$$

Conversely, (7.1) holds for some $y \in D$. Let $\{x_j\}$ be a sequence in D tending to x_0, and consider

$$u_j(x) = G(x_j, x).$$

Note that each u_j is positive and harmonic in $D - \{x_j\}$. Since $\{u_j(y)\}$ tends to zero, by Harnack's inequality, we see that $\{u_j\}$ tends to zero locally uniformly on D. Hence, if $\varphi \in C_0^\infty$, then we write $\varphi = U_2\psi$ and have

$$\begin{aligned}\int \varphi\ d\delta_{x_j,\partial D} &= \int U_2\psi\ d\delta_{x_j,\partial D} = \int U_2\delta_{x_j,\partial D}\ \psi\ dz \\ &= \int U_2\delta_{x_j}\ \psi\ dz - \int u_j\ \psi\ dz \\ &= \varphi(x_j) - \int u_j\ \psi\ dz.\end{aligned}$$

Applying Lebesgue's dominated convergence theorem, we have

$$\lim_{x\to x_0, x\in D} \int \varphi\ d\delta_{x_j,\partial D} = \varphi(x_0),$$

which implies that $\delta_{x_j,\partial D} \to \delta_{x_0}$ vaguely as $j \to \infty$. Hence it follows that $\delta_{x,\partial D} \to \delta_{x_0}$ vaguely as $x \to x_0$, $x \in D$.

COROLLARY 7.1. *Let E be the set of all points of ∂D which are irregular. Then $C_2(E) = 0$, that is, E is polar.*

COROLLARY 7.2. *Suppose every boundary point of D is regular. Then, for a continuous function f on ∂D, H_f is the Dirichlet solution for D and f.*

As in Theorem 6.4, we can show the following characterization of regular (or irregular) boundary points.

THEOREM 7.4. *Let $x_0 \in \partial D$. Then x_0 is irregular if and only if F is thin at x_0.*

COROLLARY 7.3. *Let $x_0 \in \partial D$. If F contains a continuum γ for which $x_0 \in \gamma$, then x_0 is regular.*

3.8 Removable singularities

For an open set G, let $\mathcal{H}(G)$ be a class of functions on G. A compact set $K \subsetneqq G$ is removable for $\mathcal{H}$ if any function in $\mathcal{H}(G)$ which is harmonic in $G - K$ is harmonic in G. In this section we consider the the Hölder space Λ_α for $\mathcal{H}$.

For a positive number α, take the nonnegative integer k for which

$$k < \alpha \leqq k + 1.$$

We denote by $\Lambda_\alpha(G)$ the Hölder space of all functions $u \in C^k(G)$ such that in case $\alpha < k + 1$,

$$|D^\mu u(x) - D^\mu u(y)| \leqq M|x - y|^{\alpha - k}$$

whenever x, $y \in G$ and $|\mu| = k$; in case $\alpha = k + 1$,

$$|D^\mu u(x + y) + D^\mu u(x - y) - 2D^\mu u(x)| \leqq M|y|$$

whenever x, $x \pm y \in G$ and $|\mu| = k$.

LEMMA 8.1. *If $u \in \Lambda_\alpha(G)$, then for any ball $B = B(x, r) \subsetneqq B(x, 2r) \subsetneqq G$ there exists a polynomial P_B of degree at most α such that*

$$\frac{1}{|B|} \int_B |u(y) - P_B(y)| dy \leqq A r^{\alpha - k},$$

where A is a positive constant independent of B.

PROOF. We show this only when $\alpha = 1$. Take a mollifier $\psi \in C_0^\infty(\mathbf{R}^n)$ such that $\int \psi(x) dx = 1$, $\psi = 0$ outside $B(0, 1)$ and $0 \leqq \psi \leqq 1$ on $\mathbf{R}^n$. Letting $\psi_r(y) = r^{-n}\psi(y/r)$, we consider $u_r = u * \psi_r$. Then

$$\int_B |u(y) - u_r(y)| dy \leqq 2^{-1} \sup_{z \in B(0,r)} \int_B |2u(y) - u(y - z) - u(y + z)| dy \leqq M r^{1+n}.$$

Further, we have

$$\begin{aligned}|\nabla^2 u_r(y)| &= |u * (\nabla^2 \psi_r)(y)| \\ &\leqq Mr^{-2-n} \int_{B(0,r)} |2u(y) - u(y-z) - u(y+z)|dz \leqq Mr^{-1},\end{aligned}$$

so that Taylor's theorem implies that $|u_r(y) - P(y)| \leqq Mr$ for all $y \in B$, where $P(y) = u_r(x) + (y-x) \cdot \nabla u_r(x)$. Now it follows that

$$\frac{1}{|B|} \int_B |u(y) - P(y)|dy \leqq Mr,$$

as required.

LEMMA 8.2 (Partition of unity). *Let* $\{B(x_j, r_j)\}$, $1 \leqq j \leqq N$, *be a finite family of open balls such that* $\{B(x_j, r_j/5)\}$ *is mutually disjoint. Then there exists a family* $\{\psi_j\} \subseteqq C_0^\infty(\mathbf{R}^n)$ *with the following properties :*

(i) $\psi_j \geqq 0$ *on* $\mathbf{R}^n$ *and* $\psi_j = 0$ *outside* $B(x_j, 2r_j)$.

(ii) $\sum_{j=1}^N \psi_j \leqq 1$ *on* $\mathbf{R}^n$ *and* $\sum_{j=1}^N \psi_j = 1$ *on* $\bigcup_{j=1}^N B(x_j, r_j)$.

(iii) $|D^\mu \psi_j| \leqq M_\mu r_j^{-|\mu|}$ *on* $\mathbf{R}^n$ *for any multi-index* μ.

PROOF. Without loss of generality, we may assume that $r_1 \geqq r_2 \geqq \cdots \geqq r_N$. First take a function $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that $\varphi = 1$ on $\mathbf{B}$, $\varphi = 0$ outside $B(0,2)$ and $0 \leqq \varphi \leqq 1$ on $\mathbf{R}^n$. Letting $\varphi_j(x) = \varphi((x - x_j)/r_j)$, we define $\psi_1 = \varphi_1$ and

$$\psi_k = \varphi_k \prod_{j=1}^{k-1} (1 - \varphi_j)$$

for $k = 2$, 3,... . Then $\psi_k \in C_0^\infty(\mathbf{R}^n)$ and $\psi_k = 0$ outside $B(x_k, 2r_k)$. By induction we see that

$$\sum_{j=1}^k \psi_j = 1 - \prod_{j=1}^k (1 - \varphi_j),$$

which shows that

$$\sum_{j=1}^k \psi_j = 1 \qquad \text{for } x \in \bigcup_{j=1}^k B(x_j, r_j).$$

For a multi-index $\mu = (\mu_1, ..., \mu_n)$, note that

$$|D^\mu \psi_k(x)| \leqq M \sum_{\nu_1,\ldots,\nu_n} \left(\sum_{j_1=1}^k |D^{\nu_1} \varphi_{j_1}(x)| \right) \times \cdots \times \left(\sum_{j_n=1}^k |D^{\nu_n} \varphi_{j_n}(x)| \right)$$

where $\nu_1,\ldots,\nu_n$ are multi-indices such that $\nu_1+\cdots+\nu_n=\mu$. Note also that for each positive integer m, $\{j : 2^{m-1}r_k \leqq r_j < 2^m r_k \text{ and } x \in B(x_j, 2r_j)\}$ has at most A members, where A depends only on the dimension n, since $\{B(x_j, r_j/5)\}$ is mutually disjoint. Hence we have

$$\sum_{j=1}^{k} |D^{\nu_i}\varphi_j(x)| \leqq \sum_{j=1}^{k} M r_j^{-|\nu_i|} \leqq MA \sum_{m=1}^{\infty} (2^{m-1}r_k)^{-|\nu_i|} \leqq M r_k^{-|\nu_i|},$$

so that

$$|D^\mu \psi_k(x)| \leqq M r_k^{-|\nu_1|-\cdots-|\nu_n|} = M r_k^{-|\mu|}.$$

THEOREM 8.1. *Let h be a measure function on $[0,\infty)$ and let u be a locally integrable function on G such that*

$$F(u) \equiv \sup_{B(x,r)} r^{-2}h(r)^{-1} \inf_{v} \int_{B(x,r)} |u(y)-v(y)|dy < \infty, \tag{8.1}$$

where the infimum is taken over all superharmonic functions v on $B(x,r)$. Consider the set $S(u)$ of all $x \in G$ such that

$$\limsup_{r\to 0} r^{-2-n} \int_{B(x,r)} |u(y)-v(y)|dy > 0$$

for some superharmonic function v on a neighborhood of x. If $H_h(S(u))=0$, then u can be corrected on a set of measure zero to be superharmonic in G.

PROOF. We shall show that $\Delta u \leqq 0$ on G in the sense of distributions, that is,

$$\int_G u(x)\Delta\varphi(x)dx \leqq 0 \tag{8.2}$$

for any nonnegative $\varphi \in C_0^\infty(G)$. Since $H_h(S(u))=0$ by assumption, for any $\varepsilon>0$ there exists $\{B(x_j,r_j)\}$ such that $\bigcup_j B(x_j,r_j) \supseteqq S(u)$ and $\sum_j h(r_j) < \varepsilon$. Further, for each $x \in G-S(u)$, there exist $r(x)>0$ and a superharmonic function v_x on $B(x,2r(x))$ such that

$$\int_{B(x,2r(x))} |u(y)-v_x(y)|dy < \varepsilon r(x)^{2+n}.$$

Let $\varphi \in C_0^\infty(G)$ be nonnegative. Then

$$S_\varphi \subseteqq \left(\bigcup_j B(x_j,r_j)\right) \cup \left(\bigcup_{x\in G-S(u)} B(x,r(x))\right),$$

we can choose a finite covering $\{B_j\}$ of S_φ such that $\{B_j\}=\{B_{j'}\}\cup\{B_{j''}\}$, where $B_{j'}=B(x_{j'},r_{j'})$, $B_{j''}=B(x_{j''},r_{j''})$ with $r_{j''}=r(x_{j''})$, $x_{j''}\in G-S(u)$, and $\{5^{-1}B_j\}$ is mutually disjoint. Now take $\{\psi_j\}$ for $\{B_j\}$ as in Lemma 8.2. Then

$$\begin{aligned}\int u(x)\Delta(\psi_{j'}\varphi)(x)dx &\leqq \int [u(x)-v(x)]\Delta(\psi_{j'}\varphi)(x)dx \\ &\leqq \int |u(x)-v(x)||\Delta(\psi_{j'}\varphi)(x)|dx \\ &\leqq Mh(r_{j'})[F(u)+1]\end{aligned}$$

for some v which is superharmonic in $2B_{j'}$. Similarly, we have

$$\begin{aligned}\int u(x)\Delta(\psi_{j''}\varphi)(x)dx &\leqq \int [u(x)-v_{x_{j''}}(x)]\Delta(\psi_{j''}\varphi)(x)dx \\ &\leqq \int |u(x)-v_{x_{j''}}(x)||\Delta(\psi_{j''}\varphi)(x)|dx \leqq M\varepsilon[r_{j''}]^n.\end{aligned}$$

Now it follows that

$$\begin{aligned}\int u(x)\Delta\varphi(x)dx &= \sum_j \int u(x)\Delta(\psi_j\varphi)(x)dx \\ &\leqq M\sum_{j'} h(r_{j'}) + M\varepsilon\sum_{j''}[r_{j''}]^n \leqq M\varepsilon,\end{aligned}$$

which shows (8.2), as required.

COROLLARY 8.1. *Let K be a compact subset of G, and $0<\alpha<2$. If $u\in\Lambda_\alpha(G)$ is harmonic in $G-K$ and $H_{n-2+\alpha}(K)=0$, then u is harmonic in G, that is, K is removable for $\Lambda_\alpha(G)$.*

In fact, let $h(r)=r^{n-2+\alpha}$. If $0<\alpha<1$, then the constant function is harmonic and

$$\begin{aligned}r^{-2}h(r)^{-1}\inf_v \int_{B(x,r)} |u(y)-v(y)|dy &\leqq Mr^{-2}h(r)^{-1}\int_{B(x,r)} |y-x|^\alpha dy \\ &= M(n+\alpha)^{-1}\omega_n<\infty;\end{aligned}$$

in case $1\leqq\alpha<2$, with the aid of Lemma 8.1, we see that (8.1) holds. Since $S(u)\subseteqq K$, we apply Theorem 8.1 with $\pm\, u$ to obtain the required assertion.

COROLLARY 8.2. *Let K be a compact subset of G such that $H_{n-2}(K)=0$. If u is harmonic in $G-K$ and in $BMO(G)$, that is,*

$$\sup_B \frac{1}{|B|}\int_B |u(y)-u_B|dy<\infty,$$

where the supremum is taken over all balls B in G and $u_B=\dfrac{1}{|B|}\displaystyle\int_B u(y)dy$, then u can be corrected on a set of measure zero to be harmonic in G.

We discuss the converse of Corollary 8.1.

PROPOSITION 8.1. *Let K be a compact subset of $\mathbf{R}^n$ such that $H_{n-2+\alpha}(K) > 0$, where $0 < \alpha < 2$. Then there exists $u \in \Lambda_\alpha(\mathbf{R}^n)$ such that u is harmonic in $\mathbf{R}^n - K$ and u is not harmonic in $\mathbf{R}^n$.*

PROOF. In view of Frostman's theorem, we can find a nonnegative measure μ on K such that $\mu(K) > 0$ and

$$\mu(B(x,r)) \leqq r^{n-2+\alpha} \qquad \text{for all } B(x,r). \tag{8.3}$$

In case $n > 2$, consider the potential

$$u(x) = \int |x-z|^{2-n} d\mu(z).$$

Since $S_\mu \subseteqq K$, u is harmonic outside K. If $|j| < \alpha$, then (8.3) yields

$$\begin{aligned}
\int_{B(a,r)} |D^j N(x-z)| d\mu(z) &\leqq M \int_{B(a,r)} |x-z|^{2-|j|-n} d\mu(z) \\
&\leqq M \int_{B(x,r)} |x-z|^{2-|j|-n} d\mu(z) \\
&\quad + M \int_{B(a,r)} |a-z|^{2-|j|-n} d\mu(z) \leqq M r^{\alpha-|j|}
\end{aligned}$$

for every ball $B(a,r)$. On the other hand, if $\alpha = |j|+1$ and $r = 2|y|$, then

$$\begin{aligned}
&\int_{\mathbf{R}^n - B(x,r)} |D^j N(x+y-z) + D^j N(x-y-z) - 2D^j N(x-z)| d\mu(z) \\
\leqq\ & M \int_{\mathbf{R}^n - B(x,r)} |x-z|^{-|j|-n} d\mu(z) \leqq M r^{\alpha-|j|};
\end{aligned}$$

if $\alpha < |j|+1$ and $r = 2|y|$, then

$$\begin{aligned}
&\int_{\mathbf{R}^n - B(x,r)} |D^j N(x+y-z) - D^j N(x-z)| d\mu(z) \\
\leqq\ & M \int_{\mathbf{R}^n - B(x,r)} |x-z|^{1-|j|-n} d\mu(z) \leqq M r^{\alpha-|j|}.
\end{aligned}$$

Thus it follows that $u \in \Lambda_\alpha(\mathbf{R}^n)$.

What remains is to show the case $n = 2$. In view of (8.3), μ has no point mass, so that there exist two disjoint compact subsets K_1 and K_2 of K such that $\mu(K_i) > 0$ for $i = 1, 2$. Set

$$\nu = \frac{\mu|_{K_1}}{\mu(K_1)} - \frac{\mu|_{K_2}}{\mu(K_2)} \qquad \text{and} \qquad u(x) = \int \log \frac{1}{|x-z|}\, d\nu(z).$$

Since $\nu(\mathbf{R}^n) = 0$,

$$u(x) = \int \log \frac{r}{|x-z|}\, d\nu(z)$$

for any $r > 0$. Hence, noting that $|\nu|(B(x,r)) \leqq Mr^\alpha$, we have

$$\left| \int_{B(a,r)} D^j \log(r/|x-z|)\, d\nu(z) \right| \leqq Mr^{\alpha-|j|}$$

for $|j| < \alpha$. Since the integration over $\mathbf{R}^n - B(x,r)$ with $r = 2|y|$ can be estimated as in the first case, it follows that $u \in \Lambda_\alpha(\mathbf{R}^n)$.

Finally we treat the case $\alpha = 0$.

THEOREM 8.2. *Let K be a compact subset of G for which $C_2(K) = 0$. If u is bounded and harmonic in $G - K$, then u can be extended to a harmonic function on G.*

PROOF. Since $C_2(K) = 0$, there exists a measure μ with compact support such that $U_2\mu = \infty$ on K. For any $\varepsilon > 0$, $u_\varepsilon = u + \varepsilon U_2\mu$ is superharmonic outside K. If we define $u_\varepsilon = \infty$ on K, then u_ε is lower semicontinuous on G and it has super-mean-value property at each point of G. Thus u_ε is superharmonic in G. Now define

$$U(x) = \liminf_{j\to\infty}\ u_{1/j}(x)$$

and

$$\overline{U}(x) = \liminf_{y\to x}\ U(y)$$

for $x \in G$. Then $\overline{U} = u$ outside K by the continuity of u. Further $\overline{U}$ is lower semicontinuous on G. On the other hand, Fatou's lemma implies that

$$\begin{aligned} \frac{1}{|B|}\int_B \overline{U}(z)dz &= \frac{1}{|B|}\int_B U(z)dz = \frac{1}{|B|}\int_B \liminf_{j\to\infty}\ u_{1/j}(z)\ dz \\ &\leqq \liminf_{j\to\infty}\ \frac{1}{|B|}\int_B u_{1/j}(z)\ dz \leqq \liminf_{j\to\infty}\ u_{1/j}(y) = U(y) \end{aligned}$$

for any ball $B = B(y,r)$ with closure in G. Since $\overline{U}$ is essentially bounded, by letting $y \to x$, we have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \overline{U}(z)dz \leqq \overline{U}(x)$$

whenever $\overline{B(x,r)} \subseteqq G$. Thus $\overline{U}$ is superharmonic in G, and hence it is a superharmonic extension of u to G. Similarly, $-u$ also has a superharmonic extension $\overline{V}$ on G. Then $\overline{U} + \overline{V}$ is superharmonic in G. Moreover, $\overline{U} + \overline{V}$ vanishes outside K, and hence it vanishes almost everywhere on G. Thus $\overline{U} = -\overline{V}$ on G, each of which is a harmonic extension of u to G.

In view of the above proof, we have the following result.

THEOREM 8.3. *Let $x_0 \in G$ and u be harmonic in $G - \{x_0\}$. If*

$$\lim_{x\to x_0}\ N(x-x_0)u(x) = 0,$$

then u can be extended to a harmonic function on G.

Chapter 4

Riesz potentials of functions in L^p

In this chapter we study several properties of potentials of functions in L^p. Our main aim is to give a proof of Sobolev's inequality. There are various ways of proving Sobolev's inequality, and we show a proof of applying the Marcinkiewicz interpolation theorem.

4.1 The Marcinkiewicz interpolation theorem

We introduce a notion of nonincreasing rearrangements of functions. First let f be a measurable function on $\mathbf{R}^n$. Define

$$m_f(r) = |\{x : |f(x)| > r\}|$$

and

$$f^*(t) = \inf\{r \geqq 0 : m_f(r) \leqq 2|t|\}.$$

Then it is easy to note that f^* is nonincreasing and

$$f^*(m_f(r)/2) \leqq r. \tag{1.1}$$

Since m_f is continuous from the right, we see that

$$m_f(f^*(t)) \leqq 2|t|. \tag{1.2}$$

LEMMA 1.1. *f^* is continuous from the right on $[0, \infty)$.*

PROOF. First note that $f^*(t+) \equiv \sup_{\{s:s>t\}} f^*(s) \leqq f^*(t)$. Further, (1.2) gives

$$m_f(f^*(t+)) \leqq m_f(f^*(t+s)) \leqq 2(t+s) \qquad \text{whenever } t \geqq 0 \text{ and } s > 0.$$

Hence, $m_f(f^*(t+)) \leqq 2t$, which proves $f^*(t) \leqq f^*(t+)$. Thus it follows that

$$f^*(t+) = f^*(t).$$

LEMMA 1.2. *$m_{f^*}(r) = m_f(r)$ for all $r \geqq 0$.*

PROOF. Since f^* is nonincreasing on $[0, \infty)$,

$$m_{f^*}(r) = 2 \sup\{t \geqq 0 : f^*(t) > r\}. \tag{1.3}$$

Hence (1.1) gives

$$m_f(r) \geqq m_{f^*}(r).$$

Conversely, if $2|t| > m_{f^*}(r)$, then $f^*(t) \leqq r$, so that (1.2) gives

$$m_f(r) \leqq m_f(f^*(t)) \leqq 2|t|.$$

Thus, $m_f(r) \leqq m_{f^*}(r)$ by letting $2|t| \to m_{f^*}(r)$, and the proof is completed.

LEMMA 1.3. *Let $\{f_j\}$ be a sequence of measurable functions on $\mathbf{R}^n$ such that $|f_j(x)|$ increases to $f(x)$ for each $x \in \mathbf{R}^n$. Then m_{f_j} and f_j^* increase to m_f and f^*, respectively.*

PROOF. Clearly, $m_{f_j} \leqq m_f$, and Lebesgue's monotone convergence theorem implies that

$$m_f(r) = \lim_{j \to \infty} m_{f_j}(r).$$

Further, $f_j^*(t) \leqq f_{j+1}^*(t) \leqq f^*(t)$. If we set $g(t) = \lim_{j \to \infty} f_j^*(t)$, then $g(t) \leqq f^*(t)$. Note that

$$m_f(g(t)) = \lim_{j \to \infty} m_{f_j}(g(t)) \leqq \lim_{j \to \infty} m_{f_j}(f_j^*(t)) \leqq 2|t|,$$

which implies that $f^*(t) \leqq g(t)$. Thus it follows that

$$f^*(t) = g(t),$$

as required.

THEOREM 1.1. *If $1 \leqq p < \infty$, then*

$$\int_{\mathbf{R}^n} |f(x)|^p dx = \int_{\mathbf{R}^1} f^*(t)^p dt. \tag{1.4}$$

PROOF. In view of Lemma 1.2, we have

$$\int |f(x)|^p dx = \int m_f(r) d(r^p) = \int m_{f^*}(r) d(r^p) = \int f^*(t)^p dt.$$

THEOREM 1.2. *If k is a nonnegative and nonincreasing function on $(0, \infty)$, then*

$$\int k(|x-y|)|f(y)|dy \leqq \int k(|t|) f^*(t) dt.$$

Proof. If f is a characteristic function of a set I with measure 2ℓ, that is, $f = \chi_I$, then

$$\int k(|x-y|)f(y)dy = \int_I k(|x-y|)dy \leqq \int_{-\ell}^{\ell} k(|t|)dt = \int k(|t|)f^*(t)dt.$$

Let $f = \sum_{j=1}^{m} a_j \chi_{I_j}$, where $0 < a_1 < a_2 < \cdots < a_m$ and $\{I_j\}$ are mutually disjoint. Then, writing $b_i = a_i - a_{i-1}$ with $a_0 = 0$ and $J_i = I_i \cup I_{i+1} \cup \cdots \cup I_m$, we see that

$$f = \sum_{i=1}^{m} b_i \chi_{J_i} \quad \text{and} \quad f^* = \sum_{i=1}^{m} b_i \chi_{J_i}^*.$$

Hence, by the above considerations, we have

$$\begin{aligned} \int k(|x-y|)f(y)dy &= \sum_{j=1}^{m} b_i \int_{J_i} k(|x-t|)dt \\ &\leqq \sum_{i=1}^{m} b_i \int k(|t|)\chi_{J_i}^*(t)dt = \int k(|t|)f^*(t)dt. \end{aligned}$$

Now the general case follows from considering a sequence $\{f_j\}$ of step functions which increases to f.

We here prepare the well-known Hardy's inequalities, which is an easy consequence of the following result.

Lemma 1.4. *Let $K(x,y)$ be a nonnegative measurable function on $\mathbf{R}_+ \times \mathbf{R}_+$ with $\mathbf{R}_+ = (0,\infty)$, which is homogeneous of degree -1, that is,*

$$K(\lambda x, \lambda y) = \lambda^{-1} K(x,y) \qquad \text{for all } \lambda, x, y \in \mathbf{R}_+.$$

Further

$$A_K \equiv \int_0^\infty K(1,y)y^{-1/p}dy < \infty. \tag{1.5}$$

For a nonnegative measurable function f on $\mathbf{R}_+$, set

$$Kf(x) = \int_0^\infty K(x,y)f(y)dy.$$

Then we have for $1 \leqq p \leqq \infty$

$$\|Kf\|_p \leqq A_K \|f\|_p.$$

Proof. Note by the homogeneity that

$$Kf(x) = \int_0^\infty K(1,y)f(xy)dy.$$

Now we apply Minkowski's inequality for integral to have

$$\begin{aligned} \|Kf\|_p &\leqq \int_0^\infty K(1,y)\|f(\cdot y)\|_p dy \\ &= \left(\int_0^\infty K(1,y)y^{-1/p}dy\right)\|f\|_p = A_K\|f\|_p. \end{aligned}$$

THEOREM 1.3 (Hardy's inequalities). *If f is a nonnegative measurable function f on $\mathbf{R}_+$ and $r > 0$, then*

$$\left\{\int_0^\infty \left(\int_0^x f(y)dy\right)^p x^{-r-1}dx\right\}^{1/p} \leqq p/r\left(\int_0^\infty [yf(y)]^p y^{-r-1}dy\right)^{1/p}$$

and

$$\left\{\int_0^\infty \left(\int_x^\infty f(y)dy\right)^p x^{r-1}dx\right\}^{1/p} \leqq p/r\left(\int_0^\infty [yf(y)]^p y^{r-1}dy\right)^{1/p}.$$

To prove the first inequality, consider

$$K(x,y) = \begin{cases} x^{-(r+1)/p}y^{-1+(r+1)/p} & \text{when } 0 < y < x, \\ \\ 0 & \text{when } y \geqq x \end{cases}$$

Then K satisfies all the conditions on K required in Lemma 1.4

Let $1 \leqq p_j \leqq q_j \leqq \infty$, $j = 0, 1$, $p_0 < p_1$ and $q_0 \neq q_1$. Let T be a quasi-linear transformation which is defined on $L^{p_0}(\mathbf{R}^n) + L^{p_1}(\mathbf{R}^n)$, that is,

$$|T(f_1 + f_2)(x)| \leqq \kappa(|Tf_1(x)| + |Tf_2(x)|)$$

holds almost everywhere, where κ is a positive constant independent of f_1 and f_2. We say that T is of weak type (p,q) if

$$|\{x : |Tf(x)| > \lambda\}| \leqq M\left(\frac{\|f\|_p}{\lambda}\right)^q$$

whenever $f \in L^p(\mathbf{R}^n)$ and $\lambda > 0$; we say that T is of (strong) type (p,q) if

$$\|Tf\|_q \leqq M\|f\|_p \qquad \text{for all } f \in L^p(\mathbf{R}^n).$$

Clearly, if T is of type (p,q), then T is of weak type (p,q).

THEOREM 1.4 (the Marcinkiewicz interpolation theorem). *Suppose T is of weak type (p_j, q_j) for $j = 0$ and 1, that is,*

$$|\{x : |Tf(x)| > \lambda\}| \leqq \left(\frac{A_j\|f\|_{p_j}}{\lambda}\right)^{q_j}$$

for $f \in L^{p_j}(\mathbf{R}^n)$ and $\lambda > 0$, $j = 0, 1$. If $0 < \theta < 1$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then T is of type (p, q), that is,

$$\|Tf\|_q \leqq A\|f\|_p \qquad \text{with } A = A(p_0, p_1, q_0, q_1, \theta).$$

Proof. Set

$$m(t) = |\{x : |Tf(x)| > t\}|$$

and

$$h^*(r) = \inf\{t \geqq 0 : m(t) \leqq 2|r|\}.$$

Then we have by Theorem 1.1 that

$$\|h^*\|_p = \left(\int_{-\infty}^{\infty} [h^*(r)]^p dr\right)^{1/p} = \left(\int_{\mathbf{R}^n} |h(x)|^p dx\right)^{1/p} = \|h\|_p. \tag{1.6}$$

On the other hand, it is useful to note that

$$\left(\int_0^{\infty} [r^{1/p}h^*(r)]^{q_2} dr/r\right)^{1/q_2} \leqq M\left(\int_0^{\infty} [r^{1/p}h^*(r)]^{q_1} dr/r\right)^{1/q_1} \tag{1.7}$$

for $0 < p \leqq \infty$, where $1 \leqq q_1 \leqq q_2 \leqq \infty$. To show this, we see that

$$\int_0^{\infty} [r^{1/p}h^*(r)]^{q_1} dr/r \geqq \int_{t/2}^{t} [r^{1/p}h^*(r)]^{q_1} dr/r \geqq M[t^{1/p}h^*(t)]^{q_1}$$

for $t > 0$, which proves the case $q_2 = \infty$. In case $q_1 \leqq q_2 < \infty$,

$$\int_0^{\infty} [r^{1/p}h^*(r)]^{q_2} dr/r \leqq \left(\sup_{t>0} [t^{1/p}h^*(t)]^{q_2-q_1}\right) \int_0^{\infty} [r^{1/p}h^*(r)]^{q_1} dr/r.$$

Let

$$\sigma = \frac{1/q_0 - 1/q}{1/p_0 - 1/p} = \frac{1/q - 1/q_1}{1/p - 1/p_1}.$$

For $f \in L^p(\mathbf{R}^n)$, write $f = f^t + f_t$, where

$$f^t(x) = \begin{cases} f(x) & \text{when } |f(x)| > f^*(t^\sigma), \\ 0 & \text{otherwise} \end{cases}$$

and $f_t = f - f^t$. We see that

$$\begin{aligned} &(f^t)^*(y) \leqq f^*(y) && \text{when } 0 \leqq y \leqq t^\sigma, \\ &(f^t)^*(y) = 0 && \text{when } y > t^\sigma, \\ &(f_t)^*(y) \leqq f^*(t^\sigma) && \text{when } 0 \leqq y \leqq t^\sigma, \\ &(f_t)^*(y) \leqq f^*(y) && \text{when } y \geqq t^\sigma. \end{aligned}$$

Since T is of weak type (p_j, q_j), $j = 0, 1$, we have

$$(Tf)^*(t) \leqq A_j t^{-1/q_j} \|f\|_{p_j}.$$

Moreover, since $f \in L^p(\mathbf{R}^n)$ and $p_0 < p < p_1$, it follows that $f^t \in L^{p_0}(\mathbf{R}^n)$ and $f_t \in L^{p_1}(\mathbf{R}^n)$. Now we find

$$\begin{aligned}(Tf)^*(t) &\leqq \kappa[(Tf^t)^*(t/2) + (Tf_t)^*(t/2)] \\ &\leqq \kappa\left[A_0(t/2)^{-1/q_0}\|f^t\|_{p_0} + A_1(t/2)^{-1/q_1}\|f_t\|_{p_1}\right].\end{aligned}$$

In view of (1.7), we obtain

$$\begin{aligned}\|Tf\|_q &= \|(Tf)^*\|_q = \left(2\int_0^\infty [r^{1/q}(Tf)^*(r)]^q dr/r\right)^{1/q} \\ &\leqq M\left(\int_0^\infty [r^{1/q}(Tf)^*(r)]^p dr/r\right)^{1/p}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|f^t\|_{p_0} &= \left(2\int_0^\infty [y^{1/p_0}(f^t)^*(y)]^{p_0} dy/y\right)^{1/p_0} \\ &\leqq M\int_0^\infty y^{1/p_0}(f^t)^*(y)dy/y \\ &\leqq M\int_0^{t^\sigma} y^{1/p_0} f^*(y)dy/y\end{aligned}$$

and, similarly,

$$\|f_t\|_{p_1} \leqq Mt^{\sigma/p_1} f^*(t^\sigma) + M\int_{t^\sigma}^\infty y^{1/p_1} f^*(y)dy/y.$$

Thus it follows from Hardy's inequalities that

$$\begin{aligned}\|Tf\|_q &\leqq M\left(\int_0^\infty [t^{1/q-1/q_0}\|f^t\|_{p_0}]^p dt/t\right)^{1/p} + M\left(\int_0^\infty [t^{1/q-1/q_1}\|f_t\|_{p_1}]^p dt/t\right)^{1/p} \\ &\leqq M\left\{\int_0^\infty \left(t^{1/q-1/q_0}\int_0^{t^\sigma} y^{1/p_0}(f^t)^*(y)dy/y\right)^p dt/t\right\}^{1/p} \\ &\quad + M\left(\int_0^\infty [t^{1/q-1/q_1+\sigma/p_1} f^*(t^\sigma)]^p dt/t\right)^{1/p} \\ &\quad + M\left\{\int_0^\infty \left(t^{1/q-1/q_1}\int_{t^\sigma}^\infty y^{1/p_1}(f_t)^*(y)dy/y\right)^p dt/t\right\}^{1/p} \\ &\leqq M\left\{\int_0^\infty \left(r^{1/p-1/p_0}\int_0^r y^{1/p_0} f^*(y)dy/y\right)^p dr/r\right\}^{1/p} + M\|f^*\|_p \\ &\quad + M\left\{\int_0^\infty \left(r^{1/p-1/p_1}\int_r^\infty y^{1/p_1} f^*(y)dy/y\right)^p dr/r\right\}^{1/p} \\ &\leqq M\|f^*\|_p = M\|f\|_p.\end{aligned}$$

For a locally integrable function f, we recall the definition of maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

Hölder's inequality gives

$$Mf(x) \leqq [M(|f|^p)(x)]^{1/p}.$$

Hence in view of Theorem 10.2 in Chapter 1, we see that the mapping $f \to Mf$ is of weak type (p,p) whenever $1 < p < \infty$.

COROLLARY 1.1. *If $1 < p < \infty$, then*

$$\|Mf\|_p \leqq A_p \|f\|_p$$

for every $f \in L^p(\mathbf{R}^n)$.

This can also be proved by a direct application of Theorem 10.2 in Chapter 1.

4.2 Sobolev's inequality

In case $0 < \alpha < n$, we consider the Riesz potential $U_\alpha f$ of order α for a measurable function f, which is in fact defined by

$$U_\alpha f(x) = U_\alpha * f(x) = \int |x-y|^{\alpha-n} f(y)\ dy;$$

this is also called the α-potential of f. Here we assume that $U_\alpha |f| \not\equiv \infty$, which is equivalent to

$$\int (1+|y|)^{\alpha-n} |f(y)|\ dy < \infty \tag{2.1}$$

in view of Theorem 1.1 in Chapter 2. If this is the case, then we see easily that $U_\alpha f$ is locally integrable on $\mathbf{R}^n$. This fact can be extended in the following manner, which is widely known as Sobolev's theorem.

THEOREM 2.1. *Let $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{\alpha}{n}$.*

(i) *If $1/q > 1/p^*$ and $q > 0$, then*

$$\left(\int_K |U_\alpha f(x)|^q\ dx \right)^{1/q} \leqq M_K \|f\|_p$$

for any compact set K.

(ii) *If $1 < p < \infty$ and $1/p^* > 0$, then*

$$\left(\int |U_\alpha f(x)|^{p^*} \, dx \right)^{1/p^*} \leqq M \|f\|_p. \tag{2.2}$$

REMARK 2.1. Inequality (2.2) is known as Sobolev's inequality.

Our proof can be carried out by an application of the Marcinkiewicz interpolation theorem.

PROOF OF THEOREM 2.1. We may assume that f is nonnegative. For $R > 0$, write

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x,2R)} |x-y|^{\alpha-n} f(y) \, dy + \int_{\mathbf{R}^n - B(x,2R)} |x-y|^{\alpha-n} f(y) \, dy \\ &= u_1(x) + u_2(x). \end{aligned}$$

If $x \in B(0,R)$, then

$$u_2(x) \leqq \int_{\mathbf{R}^n - B(0,R)} (|y|/2)^{\alpha-n} f(y) \, dy,$$

which implies that u_2 is bounded on $B(0,R)$. Let $0 < \delta < 1$ and

$$\delta(n-\alpha)p' < n. \tag{2.3}$$

By Hölder's inequality we have

$$\begin{aligned} u_1(x) &\leqq \left(\int_{B(x,2R)} |x-y|^{\delta(\alpha-n)p'} \, dy \right)^{1/p'} \\ &\quad \times \left(\int_{B(x,2R)} |x-y|^{(1-\delta)(\alpha-n)p} f(y)^p \, dy \right)^{1/p} \\ &= MR^{[\delta(\alpha-n)p'+n]/p'} \left(\int_{B(x,2R)} |x-y|^{(1-\delta)(\alpha-n)p} f(y)^p \, dy \right)^{1/p}. \end{aligned}$$

Now it follows from Minkowski's inequality for integral that

$$\begin{aligned} \left(\int u_1(x)^q dx \right)^{1/q} &\leqq MR^{[\delta(\alpha-n)p'+n]/p'} \\ &\quad \times \left\{ \int \left(\int_{B(y,2R)} |x-y|^{(1-\delta)(\alpha-n)p(q/p)} \, dx \right)^{p/q} f(y)^p \, dy \right\}^{1/p} \\ &\leqq MR^{[\delta(\alpha-n)p'+n]/p' + [(1-\delta)(\alpha-n)q+n]/q} \|f\|_p, \end{aligned}$$

as long as

$$(1-\delta)(\alpha-n)q+n>0. \tag{2.4}$$

If $1/q > 1/p - \alpha/n$, then we can find δ satisfying both (2.3) and (2.4), and (i) follows.

Next let $1/q = 1/p^* = 1/p - \alpha/n > 0$. To obtain Sobolev's inequality, we return to the estimate for u_2. In fact, since $\alpha p < n$, we have by Hölder's inequality

$$\begin{aligned} u_2(x) &\leqq \left(\int_{\mathbf{R}^n - B(x,2R)} |x-y|^{(\alpha-n)p'}\, dy \right)^{1/p'} \left(\int_{\mathbf{R}^n - B(x,2R)} f(y)^p\, dy \right)^{1/p} \\ &\leqq MR^{[(\alpha-n)p'+n]/p'} \|f\|_p. \end{aligned}$$

For any $\lambda > 0$, choose $R > 0$ so that

$$MR^{[(\alpha-n)p'+n]/p'} \|f\|_p = \lambda.$$

Then it follows that

$$\begin{aligned} |\{x : U_\alpha f(x) > 2\lambda\}| &\leqq |\{x : u_1(x) > \lambda\}| \\ &\leqq \int \left(\frac{u_1(x)}{\lambda} \right)^p dx \\ &\leqq M \left(R^\alpha \lambda^{-1} \|f\|_p \right)^p \\ &= M \left(\frac{\|f\|_p}{\lambda} \right)^{p^*}. \end{aligned}$$

This implies that $f \to U_\alpha f$ is of weak type (p, p^*). In view of the Marcinkiewicz interpolation theorem, the mapping is seen to be of (strong) type (p, p^*), which means the required Sobolev inequality.

THEOREM 2.2. *Let* $0 < \alpha - \dfrac{n}{p} < 1$. *If f is a function in* $L^p(\mathbf{R}^n)$ *satisfying* (2.1), *then*

$$|U_\alpha f(x) - U_\alpha f(z)| \leqq M|x-z|^{\alpha - n/p} \|f\|_p.$$

PROOF. For $r = |x-z|$, write

$$\begin{aligned} U_\alpha f(z) &= \int_{B(x,2r)} |z-y|^{\alpha-n} f(y)\, dy + \int_{\mathbf{R}^n - B(x,2r)} |z-y|^{\alpha-n} f(y)\, dy \\ &= u_1(z) + u_2(z). \end{aligned}$$

Since $(\alpha - n)p' + n > 0$, we have by Hölder's inequality

$$\begin{aligned} |u_1(z)| &\leqq \left(\int_{B(x,2r)} |z-y|^{(\alpha-n)p'}\, dy \right)^{1/p'} \left(\int_{B(x,2r)} |f(y)|^p\, dy \right)^{1/p} \\ &\leqq \left(\int_{B(x,2r)} |x-y|^{(\alpha-n)p'}\, dy \right)^{1/p'} \|f\|_p \\ &= Mr^{[(\alpha-n)p'+n]/p'} \|f\|_p. \end{aligned}$$

On the other hand,

$$|u_2(x) - u_2(z)| \leqq Mr \int_{\mathbf{R}^n - B(x,2r)} |x-y|^{\alpha-n-1}|f(y)|\, dy.$$

Since $(\alpha - n - 1)p' + n < 0$, we have as above

$$\begin{aligned} |u_2(x) - u_2(z)| &\leqq Mr \left(\int_{\mathbf{R}^n - B(x,2r)} |x-y|^{(\alpha-n-1)p'}\, dy \right)^{1/p'} \\ &\quad \times \left(\int_{\mathbf{R}^n - B(x,2r)} |f(y)|^p\, dy \right)^{1/p} \\ &\leqq Mr^{[(\alpha-n)p'+n]/p'} \|f\|_p. \end{aligned}$$

Thus the required result follows.

THEOREM 2.3. *Let $\alpha p = n$ and G be a bounded open set in $\mathbf{R}^n$. Then there exist two positive constants A_1 and A_2 such that for any $f \in L^p(G)$,*

$$\frac{1}{|G|} \int_G \exp\left[\frac{U_\alpha f}{A_1 \|f\|_p} \right]^{p'} dx \leqq A_2.$$

PROOF. Let R be chosen so that $|G| = |B(0,R)| = \sigma_n R^n$. Then for $0 < \beta < n$,

$$\int_G |x-y|^{-\beta} dy \leqq \int_{B(x,R)} |x-y|^{-\beta} dy = \omega_n R^{-\beta+n}/(n-\beta).$$

Assume that $f \geqq 0$ on G and $1 < p < q < \infty$. Then, in view of the estimates of u_1 in the proof of Theorem 2.1, we have

$$\begin{aligned} U_\alpha f(x) &\leqq \left(\omega_n R^{\delta(\alpha-n)p'+n} / [n + \delta(\alpha-n)p'] \right)^{1/p'} \\ &\quad \times \left(\int_G |x-y|^{(1-\delta)(\alpha-n)p} f(y)^p dy \right)^{1/p} \end{aligned}$$

and, by applying Minkowski's inequality for integral,

$$\begin{aligned} \left(\int_G [U_\alpha f]^q dx \right)^{1/q} &\leqq \left(\omega_n R^{\delta(\alpha-n)p'+n} / [n + \delta(\alpha-n)p'] \right)^{1/p'} \\ &\quad \times \left(\omega_n R^{(1-\delta)(\alpha-n)q+n} / [n + (1-\delta)(\alpha-n)q] \right)^{1/q} \|f\|_p. \end{aligned}$$

Here, let $1/r = 1 - 1/p + 1/q$ and $\delta = r/p'$. Then it follows that

$$\left(\int_G [U_\alpha f]^q dx \right)^{1/q} \leqq \omega_n^{1/r} R^{(\alpha-n)+n/r} [n + r(\alpha-n)]^{-1/r} \|f\|_p.$$

Since $(\alpha - n) + n/r = n/q$ and $1/r < 1$,

$$\int_G [U_\alpha f]^q dx \leqq [K_1 q]^{q/r}[|G|/\sigma_n]\|f\|_p^q \tag{2.5}$$

with $K_1 = \omega_n/n$. If m is a positive integer such that $m > p-1$, then, by (2.5) with $q = mp'$, we have

$$\frac{1}{m!}\int_G \left(\frac{U_\alpha f}{C\|f\|_p}\right)^{p'm} dx \leqq K_1 p'[|G|/\sigma_n]\frac{m^m}{(m-1)!}\left(\frac{K_1 p'}{C^{p'}}\right)^m$$

for $C > 0$. Hence it follows that

$$\int_G \sum_{m=m_0}^{\infty} \frac{1}{m!}\left(\frac{U_\alpha f}{C\|f\|_p}\right)^{p'm} dx \leqq K_1 p'[|G|/\sigma_n]\sum_{m=m_0}^{\infty} \frac{m^m}{(m-1)!}\left(\frac{K_1 p'}{C^{p'}}\right)^m$$

for $m_0 > p-1$. The sum is convergent if $C^{p'} > eK_1 p'$. The lower term $m < m_0$ can be evaluated by (2.5).

4.3 Spherical means in L^q

For $q > 0$ and a Borel measurable function u on $\mathbf{R}^n$, consider the spherical means over the surface $S(0,r)$, which is defined by

$$S_q(u,r) = \left(\frac{1}{|S(0,r)|}\int_{S(0,r)} |u(x)|^q \, dS(x)\right)^{1/q}.$$

By a simple modification of the proof of Theorem 2.1, we have the following inequality (see also Theorem 4.2 given later).

Theorem 3.1. *Let* $\alpha p > 1$ *and* $\dfrac{1}{\tilde{p}} = \dfrac{n-\alpha p}{p(n-1)} > 0$. *Then*

$$\left(\int_{S(0,R)} |U_\alpha f(x)|^{\tilde{p}} \, dS(x)\right)^{1/\tilde{p}} \leqq M_R\|f\|_p.$$

We consider

$$\kappa(r) = \begin{cases} r^{(\alpha p - n)/p} & \alpha p < n, \\ \{\log(1/r)\}^{1/p} & \alpha p = n. \end{cases}$$

Theorem 3.2. *Let* $1 < \alpha p \leqq n$, $q > 0$ *and* $1/q > 1/\tilde{p}$. *If* f *is a function in* $L^p(\mathbf{R}^n)$ *satisfying* (2.1), *then*

$$\lim_{r \to 0} [\kappa(r)]^{-1} S_q(U_\alpha f, r) = 0.$$

PROOF. We may assume that f is nonnegative and $q > p$, because S_q is nondecreasing for q. Write for $r = |x| > 0$,

$$\begin{aligned} U_\alpha f(x) &= \int_{\mathbf{R}^n - B(0,2r)} |x-y|^{\alpha-n} f(y)\, dy + \int_{B(0,2r)} |x-y|^{\alpha-n} f(y)\, dy \\ &= u_1(x) + u_2(x). \end{aligned}$$

For any fixed $a > 0$, we see from (2.1) that $\int_{\mathbf{R}^n - B(0,a)} |x-y|^{\alpha-n} f(y)\, dy$ is bounded near the origin, so that we may assume that f vanishes outside $B(0,a)$. If $x \in S(0,r)$, then

$$\begin{aligned} u_1(x) &\leqq \int_{B(0,a)-B(0,2r)} (|y|/2)^{\alpha-n} f(y)\, dy \\ &\leqq 2^{n-\alpha} \left(\int_{B(0,a)-B(0,2r)} |y|^{(\alpha-n)p'}\, dy \right)^{1/p'} \left(\int_{B(0,a)} f(y)^p\, dy \right)^{1/p} \\ &\leqq M\kappa(r) \left(\int_{B(0,a)} f(y)^p\, dy \right)^{1/p}. \end{aligned}$$

Hence it follows that

$$\limsup_{x\to 0}\, [\kappa(|x|)]^{-1} u_1(x) \leqq M \left(\int_{B(0,a)} f(y)^p\, dy \right)^{1/p},$$

which implies that the left hand-side is zero by letting $a \to 0$. Let $0 < \delta < 1$ and

$$\frac{n-\alpha p}{p(n-\alpha)} < \delta < \frac{n-1}{q(n-\alpha)}. \tag{3.1}$$

Since $(1-\delta)(\alpha-n) + n/p' > 0$, as in the proof of Theorem 2.1, we have

$$\begin{aligned} u_2(x) &\leqq \left(\int_{B(0,2r)} |x-y|^{(1-\delta)(\alpha-n)p'}\, dy \right)^{1/p'} \\ &\quad \times \left(\int_{B(0,2r)} |x-y|^{\delta(\alpha-n)p} f(y)^p\, dy \right)^{1/p} \\ &\leqq M r^{[(1-\delta)(\alpha-n)p'+n]/p'} \left(\int_{B(0,2r)} |x-y|^{\delta(\alpha-n)p} f(y)^p\, dy \right)^{1/p}. \end{aligned}$$

Since $\delta(\alpha-n)q + n - 1 > 0$, we note that

$$S_q([U_\alpha(\cdot - y)]^\delta, r) \leqq M r^{\delta(\alpha-n)}$$

for any y. Now it follows from Minkowski's inequality for integral that

$$\begin{aligned} S_q(u_2, r) &\leqq MR^{[(1-\delta)(\alpha-n)p'+n]/p'} \\ &\quad \times \left(\int_{B(0,2r)} [S_q(U_\alpha(\cdot - y)^\delta, r)]^p f(y)^p \, dy \right)^{1/p} \\ &\leqq M\kappa(r) \left(\int_{B(0,2r)} f(y)^p \, dy \right)^{1/p}, \end{aligned}$$

which gives

$$\lim_{r \to 0} [\kappa(r)]^{-1} S_q(u_2, r) = 0.$$

Thus Theorem 3.2 is obtained.

THEOREM 3.3. *Let $q > 0$ and $\dfrac{n - \alpha p - 1}{p(n-1)} < \dfrac{1}{q} \leqq \dfrac{1}{\tilde{p}}$. If f is a function in $L^p(\mathbf{R}^n)$ satisfying* (2.1), *then*

$$\liminf_{r \to 0} [\kappa(r)]^{-1} S_q(U_\alpha f, r) = 0.$$

Before proving this fact, we note the following lemma, which is a consequence of fine limit result (see Theorem 5.3 in Chapter 2).

LEMMA 3.1. *Let $0 < \beta < 1$ and μ be a measure on the real line $\mathbf{R}$ for which $U_\beta \mu \not\equiv \infty$. Then*

$$\liminf_{r \to 0} r^{1-\beta} U_\beta \mu(r) = \mu(\{0\}).$$

PROOF. In view of Theorem 5.3 in Chapter 2, there exists a set $E \subseteqq (0, \infty)$ such that E is thin at 0 and

$$\lim_{r \to 0, r \in \mathbf{R}_+ - E} r^{1-\beta} U_\beta \mu(r) = \mu(\{0\}). \tag{3.2}$$

If $I_j = [2^{-j}, 2^{-j+1})$, then

$$2^{\beta j} C_\beta(I_j) = c$$

for some constant $c > 0$. This implies that $I_j - E_j \neq \emptyset$ for large j, which together with (3.2) implies the required assertion.

Now we give a proof of Theorem 3.3. As in the proof of Theorem 3.2, we may assume that f is nonnegative, and write $U_\alpha f = u_1 + u_2$. First we have

$$\lim_{r \to 0} [\kappa(r)]^{-1} S_q(u_1, r) = 0.$$

Let $0 < \delta < 1$ and

$$\frac{n-1}{q(n-\alpha)} < \delta < 1$$

and

$$\frac{n-\alpha p}{p(n-\alpha)} < \delta < \frac{n-1}{q(n-\alpha)} + \frac{1}{p(n-\alpha)}. \tag{3.3}$$

As in the proof of Theorem 3.2, we have

$$S_q(u_2, r) \leqq Mr^{[(1-\delta)(\alpha-n)p'+n]/p'} \left(\int_{B(0,2r)} S_q([U_\alpha(\cdot - y)]^\delta, r) f(y)^p \, dy \right)^{1/p}.$$

Set $\beta = -p[n-1-\delta(n-\alpha)q]/q$ and note that

$$S_q([U_\alpha(\cdot - y)]^\delta, r) \leqq M \left| \frac{|y|-r}{r} \right|^{-\beta}$$

for any y. Hence we see from Minkowski's inequality for integral that

$$S_q(u_2, r) \leqq M\kappa(r) \left(\int_{B(0,2r)} \left| \frac{|y|-r}{r} \right|^{-\beta} f(y)^p \, dy \right)^{1/p}.$$

Now Lemma 3.1 gives

$$\liminf_{r\to 0} \, [\kappa(r)]^{-1} S_q(u_2, r) = 0,$$

and Theorem 3.3 is proved.

4.4 Restriction property

For a point $x \in \mathbf{R}^n$, we write

$$x = (x_1, x_2, ..., x_n) = (x_1, x'), \qquad x' = (x_2, ..., x_n).$$

THEOREM 4.1. *Let* $0 < \beta = \alpha - 1/p < 1$. *Then*

$$\left(\int\int_{|x'-y'|<1} \frac{|U_\alpha f(0,x') - U_\alpha f(0,y')|^p}{|x'-y'|^{n-1+\beta p}} dx'dy' \right)^{1/p} \leqq M\|f\|_p.$$

To show Theorem 4.1, we need the following easy results.

LEMMA 4.1. *If* $a < 1$, *then*

$$\int |(z_1, z')|^{a-n} dz' = M|z_1|^{a-1}.$$

LEMMA 4.2. *If $\alpha < 2$, then*

$$\int_{\{x':|x'|>2|h'|\}} ||(z_1, x' + h')|^{\alpha-n} - |(z_1, x')|^{\alpha-n}|dx' \leqq M|h'||z_1|^{\alpha-2}.$$

In fact, note that

$$||x+h|^{\alpha-n} - |x|^{\alpha-n}| \leqq M|h||x|^{\alpha-n-1}$$

whenever $|h| \leqq 2|x|$, and apply Lemma 4.1.

PROOF OF THEOREM 4.1. Note that

$$U_\alpha f(0, x') = \int_{\mathbf{R}^1} \int_{\mathbf{R}^{n-1}} |(-z_1, x' - z')|^{\alpha-n} f(z_1, z') dz_1 dz'$$

and

$$\begin{aligned} &|U_\alpha f(0, x' + h') - U_\alpha f(0, x')| \\ \leqq \ & \int \left(\int \left| |(-z_1, x' + h' - z')|^{\alpha-n} - |(-z_1, x' - z')|^{\alpha-n} \right| |f(z_1, z')| dz' \right) dz_1. \end{aligned}$$

Hence we have by Young's inequality

$$\begin{aligned} &\|U_\alpha f(0, \cdot + h') - U_\alpha f(0, \cdot)\|_p \\ \leqq \ & \int \left(\int \left| |(-z_1, x' + h')|^{\alpha-n} - |(-z_1, x')|^{\alpha-n} \right| dx' \right) \left(\int |f(z_1, z')|^p dz' \right)^{1/p} dz_1. \end{aligned}$$

In case $\alpha < 1$, in view of Lemmas 4.1 and 4.2, we have

$$\begin{aligned} \|U_\alpha f(0, \cdot + h') - U_\alpha f(0, \cdot)\|_p &\leqq M \int_{|z_1|<|h'|} |z_1|^{\alpha-1} \|f(z_1, \cdot)\|_p dz_1 \\ &\quad + M|h'| \int_{|z_1|\geqq|h'|} |z_1|^{\alpha-2} \|f(z_1, \cdot)\|_p dz_1 \\ &= M[I_1(h') + I_2(h')]. \end{aligned}$$

By Hardy's inequality, we obtain

$$\begin{aligned} \int_{\mathbf{R}^{n-1}} \frac{[I_1(h')]^p}{|h'|^{n-1+\beta p}} dh' &\leqq M \int_0^\infty r^{-\alpha p} \left(\int_0^r |z_1|^{\alpha-1} \|f(z_1, \cdot)\|_p dz_1 \right)^p dr \\ &\leqq M \int_0^\infty |z_1|^{-\alpha p} [|z_1|^\alpha \|f(z_1, \cdot)\|_p]^p dz_1 = M\|f\|_p{}^p. \end{aligned}$$

In the same way we find

$$\begin{aligned} \int_{\mathbf{R}^{n-1}} \frac{[I_2(h')]^p}{|h'|^{n-1+\beta p}} dh' &\leqq M \int_0^\infty r^{(1-\alpha)p} \left(\int_r^\infty |z_1|^{\alpha-2} \|f(z_1, \cdot)\|_p dz_1 \right)^p dr \\ &\leqq M \int_0^\infty |z_1|^{(1-\alpha)p} [|z_1|^{\alpha-1} \|f(z_1, \cdot)\|_p]^p dz_1 = M\|f\|_p{}^p. \end{aligned}$$

Thus the case $\alpha < 1$ is proved.

In case $\alpha = 1$, we must replace I_1 by

$$\begin{aligned} J_1(h') &= \int_{|z_1|<|h'|} \log(2|h'|/|z_1|)\|f(z_1,\cdot)\|_p dz_1 \\ &\leqq M\int_{|z_1|<|h'|} [|h'|/|z_1|]^{\varepsilon}\|f(z_1,\cdot)\|_p dz_1 \end{aligned}$$

for $0 < \varepsilon < 1$ and apply Hardy's inequality. In case $1 < \alpha < 2$,

$$I_1 \leqq M|h'|\int_{|z_1|<|h'|} [|h'| + |z_1|]^{\alpha-2}\|f(z_1,\cdot)\|_p dz_1,$$

which can be treated similarly.

THEOREM 4.2. *Let $\alpha p > 1$ and $1/\tilde{p} = (n - \alpha p)/p(n-1) > 0$. Then*

$$\left(\int |U_\alpha f(0,x')|^{\tilde{p}} dx'\right)^{1/\tilde{p}} \leqq M\|f\|_p.$$

In fact, we have by Hölder's inequality

$$\begin{aligned} |U_\alpha f(0,x')| &\leqq \int_{\mathbf{R}^{n-1}} \left(\int_{\mathbf{R}^1} |(-z_1, x'-z')|^{p'(\alpha-n)} dz_1\right)^{1/p'} \\ &\quad \times \left(\int_{\mathbf{R}^1} |f(z_1,z')|^p dz_1\right)^{1/p} dz' \\ &\leqq M\int_{\mathbf{R}^{n-1}} |x'-z'|^{\alpha-n+1/p'} \left(\int_{\mathbf{R}^1} |f(z_1,z')|^p dz_1\right)^{1/p} dz'. \end{aligned}$$

Now apply Sobolev's inequality to establish the required inequality.

4.5 Inverse property

Recall from Theorem 2.9 in Chapter 2 that if $\varphi \in C_0^\infty(\mathbf{R}^n)$, then

(5.1) $$\varphi(x) = U_\alpha[\psi(\alpha,\cdot)](x);$$

for example, in case $0 < \alpha < 2$,

(5.2) $$\psi(\alpha,x) = {\gamma_\alpha}^{-1}\lim_{\varepsilon\to 0}\int_{\mathbf{R}^n - B(0,\varepsilon)} \frac{\varphi(x+t)-\varphi(x)}{|t|^{n+\alpha}} dt.$$

THEOREM 5.1. *If $0 < \alpha < 2$ and $f \in L^p(\mathbf{R}^n)$, $1 \leqq p < \infty$, then*

$$\left\|\int_{\mathbf{R}^n - B(0,\varepsilon)} \frac{U_\alpha f(\cdot+t) - U_\alpha f(\cdot)}{|t|^{n+\alpha}} dt\right\|_p \leqq M\|f\|_p$$

for any $\varepsilon > 0$*; moreover,*

$$f(y) = {\gamma_\alpha}^{-1} \lim_{\varepsilon \to 0} \int_{\mathbf{R}^n - B(0,\varepsilon)} \frac{U_\alpha f(x+t) - U_\alpha f(x)}{|t|^{n+\alpha}} dt.$$

Before giving a proof, we show that

$$K(y) = \int_{\mathbf{R}^n - \mathbf{B}} [|t-y|^{\alpha-n} - |y|^{\alpha-n}]/|t|^{n+\alpha} dt$$

is integrable on $\mathbf{R}^n$. First note that

$$(5.3) \qquad |K(y)| \leqq M|y|^{\alpha-n} \qquad \text{for } y \in B(0,1/2).$$

LEMMA 5.1. *If* $0 < \alpha < 2$ *and* $|\xi| = 1$, *then*

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^n - B(0,\varepsilon)} \frac{|t+\xi|^{\alpha-n} - 1}{|t|^{n+\alpha}} dt = 0.$$

PROOF. Consider the inversion : $s = t/|t|^2$; then $t = s/|s|^2$. We can write

$$\begin{aligned} \int_{\mathbf{R}^n - B(0,\varepsilon)} \frac{|t+\xi|^{\alpha-n} - 1}{|t|^{n+\alpha}} dt &= \int_{B(0,1/\varepsilon)} [|\xi + s/|s|^2|^{\alpha-n} - 1]|s|^{\alpha-n} ds \\ &= \int_{B(0,1/\varepsilon)} [||s|\xi + s/|s||^{\alpha-n} - |s|^{\alpha-n}] ds \\ &= \int_{B(0,1/\varepsilon)} [|s+\xi|^{\alpha-n} - |s|^{\alpha-n}] ds, \end{aligned}$$

so that

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^n - B(0,\varepsilon)} \frac{|t+\xi|^{\alpha-n} - 1}{|t|^{n+\alpha}} dt = \lim_{N \to \infty} \int_{B(0,N)} [|t+\xi|^{\alpha-n} - |t|^{\alpha-n}] dt.$$

Further we see that

$$\begin{aligned} &\int_{B(0,N)} |t+\xi|^{\alpha-n} dt = \int_{B(\xi,N)} |s|^{\alpha-n} ds \\ = &\int_{B(0,N)} |s|^{\alpha-n} ds + \int_{B(\xi,N)-B(0,N)} |s|^{\alpha-n} ds - \int_{B(0,N)-B(\xi,N)} |s|^{\alpha-n} ds \\ = &\int_{B(0,N)} |t|^{\alpha-n} dt + \int_{B(0,N)-B(\xi,N)} [|\xi - t|^{\alpha-n} - |-t|^{\alpha-n}] dt. \end{aligned}$$

Here note that

$$\begin{aligned} \left| \int_{B(0,N)-B(\xi,N)} [|\xi - t|^{\alpha-n} - |-t|^{\alpha-n}] dt \right| &\leqq MN^{\alpha-n-1} |B(0,N) - B(\xi,N)| \\ &\leqq MN^{\alpha-2}. \end{aligned}$$

Hence it follows that

$$\lim_{N\to\infty} \int_{B(0,N)} [|t+\xi|^{\alpha-n} - |t|^{\alpha-n}]dt = 0,$$

which proves the required equality.

LEMMA 5.2. *If $0<\alpha<2$, then*

$$\left| \int_{\mathbf{R}^n-\mathbf{B}} \frac{|t+y|^{\alpha-n} - |y|^{\alpha-n}}{|t|^{n+\alpha}} dt \right| \leqq M|y|^{\alpha-n-2}$$

whenever $y \in \mathbf{R}^n - \mathbf{B}$.

PROOF. We see from Lemma 5.1 that

$$\begin{aligned} & \int_{\mathbf{R}^n-\mathbf{B}} \frac{|t+y|^{\alpha-n} - |y|^{\alpha-n}}{|t|^{n+\alpha}} dt \\ = \quad & |y|^{-n} \int_{\mathbf{R}^n-B(0,1/|y|)} \frac{|s+\xi|^{\alpha-n}-1}{|s|^{n+\alpha}} ds \\ = \quad & |y|^{-n} \lim_{\varepsilon\to 0} \int_{B(0,1/|y|)-B(0,\varepsilon)} \frac{|s+\xi|^{\alpha-n}-1}{|s|^{n+\alpha}} ds \\ = \quad & |y|^{-n} \lim_{\varepsilon\to 0} \int_{B(0,1/|y|)-B(0,\varepsilon)} \frac{|s+\xi|^{\alpha-n}-1-\sum_j s_j(\partial/\partial\xi_j)|\xi|^{\alpha-n}}{|s|^{n+\alpha}} ds, \end{aligned}$$

where $\xi = y/|y|$. Hence it suffices to note that

$$\begin{aligned} \int_{B(0,1/|y|)} \frac{\left| |s+\xi|^{\alpha-n}-1-\sum_j s_j(\partial/\partial\xi_j)|\xi|^{\alpha-n} \right|}{|s|^{n+\alpha}} ds \quad & \leqq \quad M \int_{B(0,1/|y|)} |s|^{2-\alpha-n} ds \\ & \leqq \quad M|y|^{\alpha-2}. \end{aligned}$$

Lemma 5.2 together with (5.3) gives the following result.

LEMMA 5.3. *If $0<\alpha<2$, then $K \in L^1(\mathbf{R}^n)$.*

For $\varepsilon > 0$, define

$$K_\varepsilon(x) = \varepsilon^{-n} K(x/\varepsilon).$$

Now we are ready to prove Theorem 5.1.

PROOF OF THEOREM 5.1. Write

$$\int_{\mathbf{R}^n-B(x,\varepsilon)} \frac{U_\alpha f(x+t) - U_\alpha f(x)}{|t|^{n+\alpha}} dt$$

$$
\begin{aligned}
&= \int \left(\int_{\mathbf{R}^n - B(0,\varepsilon)} [|t-y|^{\alpha-n} - |y|^{\alpha-n}]/|t|^{n+\alpha} dt \right) f(x+y)dy \\
&= \int K_\varepsilon(y) f(x+y)dy.
\end{aligned}
$$

With the aid of Theorem 2.3 in Chapter 2, we see that

$$
\int K_\varepsilon(y) f(x+y)dy \to f(x) \int K(y)dy \qquad \text{in } L^p(\mathbf{R}^n).
$$

In view of (5.1) and (5.2), it now suffices to note that

$$
\int K(y)dy = \gamma_\alpha.
$$

REMARK 5.1. For general α, we need to consider the higher differences, and leave the further considerations to the reader.

Chapter 5

Continuity properties of potentials of functions in L^p

The study of (k,p)-capacities was systematically done by Meyers as generalizations of α-capacities. In this chapter we give the fundamental properties of (α,p)-capacities, and study continuity properties of potentials of functions in L^p in connection with the capacities.

5.1 (k,p)-Capacity

We say that a function k on the interval $(0,\infty)$ is a kernel if k is finite, nonnegative, nonincreasing and lower semicontinuous; further, in this book, assume

(k1) $\quad k(0) = \lim_{r \to 0} k(r) = \infty;$

(k2) $\quad k(r) \leqq Mk(2r) \qquad \text{for } r > 0;$

(k3) $\quad \int_{\mathbf{B}} k(|y|)\, dy < \infty.$

With the aid of (k3), the following is an easy modification of Theorem 1.1 in Chapter 2.

LEMMA 1.1. *Let f be a nonnegative measurable function on $\mathbf{R}^n$, and set*

$$U_k f(x) \equiv \int k(|x-y|) f(y)\, dy.$$

Then $U_k f \not\equiv \infty$ if and only if

$$\int k(1+|y|) f(y)\, dy < \infty; \tag{1.1}$$

if (1.1) *holds, then $U_k f$ is locally integrable on $\mathbf{R}^n$.*

Let $1 < p < \infty$. For a set E and an open set G, we define the relative (k,p)-capacity by

$$C_{k,p}(E;G) = \inf \ \|f\|_p^p,$$

where the infimum is taken over all nonnegative measurable functions f such that $f = 0$ outside G and $U_k f(x) \geqq 1$ for all $x \in E$. If such an f does not exist, then we set $C_{k,p}(E;G) = \infty$. In case $k(r) = r^{\alpha-n}$, we write $C_{\alpha,p}$ for $C_{k,p}$.

We first show the following result (cf. Theorem 4.1 in Chapter 2).

THEOREM 1.1. *$C_{k,p}(\cdot;G)$ is a countably subadditive, nondecreasing and outer capacity.*

PROOF. Clearly,

$$C_{k,p}(E_1;G) \leqq C_{k,p}(E_2;G) \qquad \text{whenever } E_1 \subseteqq E_2. \tag{1.2}$$

Let f be a nonnegative measurable function such that $f = 0$ outside G and $U_k f \geqq 1$ on E. Since k is lower semicontinuous, Fatou's lemma shows that $U_k f$ is also lower semicontinuous, so that $\omega(a) = \{x : U_k f > a\}$ is open for all a. Further

$$C_{k,p}(E;G) \leqq C_{k,p}(\omega(a);G) \leqq a^{-p} \int_G f(y)^p dy$$

whenever $0 < a < 1$. By letting $a \to 1$, we have

$$C_{k,p}(E;G) \leqq \inf_{\omega \supseteqq E, \omega:\text{open}} C_{k,p}(\omega;G) \leqq \int_G f(y)^p dy,$$

which gives

$$C_{k,p}(E;G) = \inf_{\omega \supseteqq E, \omega:\text{open}} C_{k,p}(\omega;G). \tag{1.3}$$

Finally, for each E_j, take a nonnegative measurable function f_j such that $f_j = 0$ outside G and $U_k f_j \geqq 1$ on E_j. Consider the function

$$f(y) = \sup_j \ f_j(y).$$

Then it is easy to see that

$$U_k f \geqq U_k f_j \geqq 1 \qquad \text{for all } x \in E_j.$$

Hence

$$C_{k,p}(\bigcup_j E_j;G) \leqq \int f(y)^p dy \leqq \sum_j \int f_j(y)^p dy,$$

which shows that

$$C_{k,p}(\bigcup_j E_j;G) \leqq \sum_j C_{k,p}(E_j;G). \tag{1.4}$$

Proposition 1.1. *For any $E \subseteq \mathbf{R}^n$, $C_{k,p}(E; G) = 0$ if and only if there exists a nonnegative function $f \in L^p(G)$ such that*

$$U_k f(x) = \infty \qquad \text{whenever } x \in E.$$

Proof. If $C_{k,p}(E; G) = 0$, then for any integer j we can find a nonnegative measurable function f_j such that $f_j = 0$ outside G, $U_k f_j \geqq 1$ on E_j and

$$\int_G f_j(y)^p dy < 2^{-j}.$$

Then $f = \sum_j f_j$ belongs to $L^p(G)$ and

$$U_k f(x) = \sum_j U_k f_j(x) = \infty \qquad \text{for any } x \in E.$$

Conversely, if there exists a nonnegative measurable function $f \in L^p(G)$ such that $U_k f(x) = \infty$ for all $x \in E$, then

$$C_{k,p}(E; G) \leqq a^{-p} \int_G f(y)^p dy$$

for every $a > 0$. By letting $a \to \infty$, we see that

$$C_{k,p}(E; G) = 0.$$

Proposition 1.2. *If $\alpha p \geqq n$, then $C_{\alpha,p}(E; \mathbf{R}^n) = 0$ for any set E.*

In fact, if $f(y) = |y|^{-\alpha}(\log |y|)^{-\delta}$ outside $B(0,2)$ for $1/p < \delta < 1$, then

$$\int_{\mathbf{R}^n - B(0,2)} f(y)^p \, dy < \infty$$

and

$$\int_{\mathbf{R}^n - B(0,2)} (1 + |y|)^{\alpha - n} f(y) \, dy = \infty.$$

The second assertion implies that

$$\int_{\mathbf{R}^n - B(0,2)} |x - y|^{\alpha - n} f(y) \, dy = \infty \qquad \text{for any } x.$$

In view of Proposition 1.2, it is convenient to say that E is of (k,p)-capacity zero, that is,

$$C_{k,p}(E) = 0$$

if $C_{k,p}(E \cap G; G) = 0$ for any bounded open set G.

PROPOSITION 1.3. *If* $\int_{\mathbf{B}} [k(|y|)]^{p'}\, dy < \infty$, *then* $C_{k,p}(E) > 0$ *whenever* E *is not empty.*

For this purpose, it suffices to note that for any x,

$$\int_{B(x,1)} k(|x-y|) f(y)\, dy \leqq \left(\int_{\mathbf{B}} [k(|y|]^{p'}\, dy \right)^{1/p'} \|f\|_p.$$

COROLLARY 1.1. *In case* $\alpha p > n$, $C_{\alpha,p}(E) = 0$ *if and only if* E *is empty.*

THEOREM 1.2. $C_{k,p}(E) = 0$ *if and only if there exists a nonnegative function* f *such that* $U_k f \not\equiv \infty$ *and*

$$U_\alpha f(x) = \infty \qquad \text{whenever } x \in E.$$

PROOF. If there exists a nonnegative function f such that $U_k f \not\equiv \infty$ and $U_k f(x) = \infty$ whenever $x \in E$, then we see that

$$\int_{B(0,j)} k(|x-y|) f(y)\, dy = \infty \qquad \text{whenever } x \in E \cap B(0,j),$$

which implies that $C_{k,p}(E \cap B(0,j); B(0,j)) = 0$. Thus $C_{k,p}(E) = 0$ follows. Conversely, suppose $C_{k,p}(E) = 0$. Then for any j, we have

$$C_{k,p}(E \cap B(0,j); B(0,j)) = 0,$$

and hence we can find f_j such that $\|f_j\|_p < 2^{-j}$,

$$\int k(1+|y|) f_j(y)\, dy < 2^{-j}$$

and

$$\int_{B(0,j)} k(|x-y|) f_j(y)\, dy = \infty \qquad \text{whenever } x \in E \cap B(0,j).$$

Now the function $f(y) = \sup_j f_j(y)$ is seen to have all the required properties.

COROLLARY 1.2. *If* $C_{k,p}(E; G) = 0$ *for a bounded open set* G, *then* $C_{k,p}(E) = 0$.

COROLLARY 1.3. *If* $\alpha p < n$ *and* $C_{\alpha,p}(E; \mathbf{R}^n) = 0$, *then* $C_{\alpha,p}(E) = 0$.

In fact, if $\alpha p < n$, then Hölder's inequality gives

$$\int (1+|y|)^{\alpha-n} |f(y)|\, dy \leqq M \|f\|_p.$$

We say that a property holds (k,p)-q.e. on a set A if it holds for all $x \in A$ except those in a set E with $C_{k,p}(E) = 0$. In case $k(r) = r^{\alpha-n}$, "(k,p)-q.e." may be written as "(α,p)-q.e." .

THEOREM 1.3. *Let G be a bounded open set in $\mathbf{R}^n$, and let $\{f_i\}$ be a sequence in $L^p(G)$ which converges to f in $L^p(G)$. Then there exists a set E and a sequence $\{i(j)\}$ such that $C_{k,p}(E;G) = 0$ and*

$$\lim_{j\to\infty} U_k f_{i(j)}(x) = U_k f(x) \qquad \text{for } (k,p)\text{-q.e. } x \in G.$$

PROOF. The proof is easier than that of Theorem 4.5 in Chapter 2. For each positive integers i and j, consider the set

$$E_{i,j} = \{x : |U_k f_i(x) - U_k f(x)| > j^{-1}\}.$$

Then we have by the definition of (k,p)-capacity

$$C_{k,p}(E_{i,j};G) \leqq j^p \int_G |f_i(y) - f(y)|^p \, dy.$$

Since $\{f_i\} \to f$ in $L^p(G)$, we can find $i(j)$ such that $i(1) < i(2) < \cdots$ and

$$\int_G |f_{i(j)}(y) - f(y)|^p \, dy \leqq 2^{-j}.$$

Set $E = \bigcap_{\ell=1}^{\infty} \left(\bigcup_{j=\ell}^{\infty} E_{i(j),j} \right)$. Then it follows from countable subadditivity that

$$C_{k,p}(E;G) \leqq \sum_{j=\ell}^{\infty} C_{k,p}(E_{i(j),j};G) \leqq \sum_{k=\ell}^{\infty} j^p 2^{-j} \to 0$$

as $\ell \to \infty$, from which $C_{k,p}(E;G) = 0$ follows. If $x \in G - E$, then we can find ℓ such that

$$|U_k f_{i(j)}(x) - U_k f(x)| \leqq j^{-1} \qquad \text{for any } j \geqq \ell,$$

which implies that

$$\lim_{j\to\infty} |U_k f_{i(j)}(x) - U_\alpha f(x)| = 0.$$

COROLLARY 1.4. *If $C_{k,p}(A;G) < \infty$, then there exists a unique nonnegative function $f \in L^p(G)$ such that $\|f\|_p^p = C_{k,p}(A;G)$ and*

$$U_k f(x) \geqq 1 \qquad \text{for } (k,p)\text{-q.e. } x \in A.$$

In fact, take a sequence $\{f_j\} \in L^p(G)$ such that $\lim_{j\to\infty} \|f_j\|_p^p = C_{k,p}(A;G)$ and

$$U_k f_j(x) \geqq 1 \qquad \text{for any } x \in A.$$

Since $\lim_{j,\ell\to\infty} \|(f_j + f_\ell)/2\|_p^p = C_{k,p}(A;G)$, we see by Clarkson's inequality that $\{f_j\}$ converges to a function f_0 in $L^p(G)$. Then $\|f_0\|_p^p = C_{k,p}(A;G)$ and, moreover, Theorem 1.3 implies that

$$U_k f_0(x) \geqq 1 \qquad \text{for } (k,p)\text{-q.e. } x \in A.$$

The uniqueness is also a consequence of Clarkson's inequality.

THEOREM 1.4. *If* $A_j \subseteqq A_{j+1}$ *and* $A = \bigcup_{j=1}^{\infty} A_j$, *then*

$$\lim_{j\to\infty} C_{k,p}(A_j;G) = C_{k,p}(A;G).$$

PROOF. We may assume that the left hand-side is finite, so that

$$C_{k,p}(A_j;G) < \gamma < \infty.$$

For each j, find f_j such that $f_j = 0$ outside G, $\|f_j\|_p^p < \gamma$ and

$$U_k f_j(x) \geqq 1 \qquad \text{whenever } x \in A_j.$$

Then there exist a subsequence $\{f_{j(\ell)}\}$ and a sequence $\{a_{i,\ell}\}$ of nonnegative numbers such that $\sum_{i\geqq\ell} a_{i,\ell} = 1$ and

$$g_\ell = \sum_{i\geqq\ell} a_{i,\ell} f_{j(i)}$$

converges to a function $f_0 \in L^p(G)$. Then we see that

$$U_k g_\ell(x) \geqq 1 \qquad \text{whenever } x \in A_{j(\ell)}.$$

Since $\|g_\ell\|_p^p < \gamma$, we apply Theorem 1.3 to obtain a subsequence $\{g_{\ell'}\}$ such that $U_k g_{\ell'} \to U_k f_0$ for every $x \in G - E$, where $C_{k,p}(E;G) = 0$. Thus

$$U_k f_0(x) \geqq 1 \qquad \text{whenever } x \in A - E.$$

Hence

$$\begin{aligned} C_{k,p}(A;G) &\leqq C_{k,p}(E;G) + C_{k,p}(A-E;G) \\ &\leqq \|f_0\|_p^p = \lim_{\ell\to\infty} \|g_\ell\|_p^p \leqq \gamma, \end{aligned}$$

which implies that

$$C_{k,p}(A;G) \leqq \lim_{j\to\infty} C_{k,p}(A_j;G).$$

Thus the equality holds.

For a subset A of an open set G, we consider the inner capacity

$$c_{k,p}(A;G) = \sup\ \mu(G),$$

where the supremum is taken over all measures μ such that $S_\mu \subseteqq A$ and

$$\|U_k\mu\|_{L^{p'}(G)} \leqq 1.$$

Then it is easy to see that

$$\begin{aligned} \mu(G) &\leqq \int U_k f(x)\ d\mu(x) \leqq \int U_k\mu(y) f(y)\ dy \\ &\leqq \|U_k\mu\|_{L^{p'}(G)} \|f\|_p \leqq \|f\|_p \end{aligned}$$

for a competing function f defining $C_{k,p}(A;G)$, so that

$$c_{k,p}(A;G) \leqq [C_{k,p}(A;G)]^{1/p}. \tag{1.5}$$

We show below that the equality in (1.5) should hold for any Borel set A.

LEMMA 1.2. *If $c_{k,p}(A;G) < \infty$, then there exists a measure μ supported by $\overline{A}$ for which*

$$\mu(\mathbf{R}^n) = c_{k,p}(A;G)$$

and

$$\|U_k\mu\|_{L^{p'}(G)} \leqq 1.$$

Recall that $\mathcal{M}_1(A)$ denotes the family of all unit measures for which $S_\mu \subseteqq A$. Further, consider

$$\mathcal{L}_1^p(G) = \{f \in L^p(G) : f \geqq 0 \ \text{ and } \ \|f\|_p \leqq 1\}.$$

LEMMA 1.3. *For $A \subseteqq G$,*

$$[c_{k,p}(A;G)]^{-1} = \inf_{\mu \in \mathcal{M}_1(A)} \ \sup_{f \in \mathcal{L}_1^p(G)} \int U_k f(x)\ d\mu(x).$$

PROOF. Set $\gamma = \inf_{\mu \in \mathcal{M}_1(A)} \|U_k\mu\|_{L^{p'}(G)}$. We have only to show that

$$\gamma = [c_{k,p}(A;G)]^{-1}, \tag{1.6}$$

because $\|g\|_{L^{p'}(G)} = \sup_{f \in \mathcal{L}_1^p(G)} \int f(y)g(y)\ dy$. If $\gamma' > \gamma$, then there exists $\mu \in \mathcal{M}_1(A)$ for which $\|U_k\mu\|_{L^{p'}(G)} < \gamma'$. Then it follows from the definition of $c_{k,p}$ that

$$c_{k,p}(A;G) \geqq \mu(\mathbf{R}^n)/\gamma' = 1/\gamma',$$

so that $\gamma \geqq [c_{k,p}(A;G)]^{-1}$.

Conversely, let μ be a competing measure for $c_{k,p}(A;G)$. If $\mu(\mathbf{R}^n) < \infty$, then

$$\gamma \leqq [1/\mu(\mathbf{R}^n)]\|U_k\mu\|_{L^{p'}(G)} \leqq 1/\mu(\mathbf{R}^n),$$

so that $\gamma \leqq [c_{k,p}(A;G)]^{-1}$. Thus (1.6) is proved.

In the same manner we can prove

LEMMA 1.4. *For $A \subseteqq G$,*

$$[C_{k,p}(A;G)]^{-p} = \sup_{f \in \mathcal{L}_1^p(G)} \inf_{\mu \in \mathcal{M}_1(A)} \int U_k f(x)\, d\mu(x).$$

Now, in view of minimax lemma, we see that

$$c_{k,p}(K;G) = [C_{k,p}(K;G)]^{1/p}$$

for any compact subset K of G. By appealing the capacitability result with the aid of Theorem 1.4, we can prove the following general result.

THEOREM 1.5. *For every Suslin set A, we have*

$$c_{k,p}(A;G) = [C_{k,p}(A;G)]^{1/p}.$$

THEOREM 1.6. *Let A be a Suslin subset of G for which $0 < C_{k,p}(A;G) < \infty$. Then there exist $f \in L^p(G)$ and $\mu \in \mathcal{M}(\overline{A})$ such that*

(i) $\|f\|_p^p = C_{k,p}(A;G)$;

(ii) $U_k f \geqq 1 \quad (k,p)$-q.e. *on A;*

(iii) $\mu(\overline{A}) = c_{k,p}(A;G)\ (= [C_{k,p}(A;G)]^{1/p}\ )$;

(iv) $\|U_k\mu\|_{L^{p'}(G)} = 1$;

(v) $\mu(\{x : U_k f(x) \neq 1\}) = 0$;

(vi) $U_k\mu(y) = [C_{k,p}(A;G)]^{-1/p'}[f(y)]^{p-1}$;

(vii) $\int_G |x-y|^{\alpha-n}[U_\alpha\mu(y)]^{1/(p-1)}dy \leqq [c_{k,p}(A;G)]^{-1}$ *on S_μ.*

PROOF. By Corollary 1.3, we can find a function f satisfying (i) and

$$U_k f(x) \geqq 1 \qquad \text{for any } x \in A - E,$$

where $C_{k,p}(E) = 0$; further take a measure μ as in Lemma 1.1. In view of (1.5), we have

$$\mu(B) \leqq \|U_k\mu\|_{L^{p'}(G)} C_{k,p}(B; G)$$

for any Borel set $B \subseteqq G$, which implies that

$$\mu(E) = 0.$$

Hence

$$\begin{aligned} c_{k,p}(A; G) &= \mu(\mathbf{R}^n) \leqq \int U_k f(x)\, d\mu(x) \\ &= \int U_k\mu(y) f(y)\, dy \\ &\leqq \|U_k\mu\|_{L^{p'}(G)} \|f\|_p \\ &\leqq \|f\|_p = [C_{k,p}(A; G)]^{1/p}. \end{aligned}$$

Since $c_{k,p}(A; G) = [C_{k,p}(A; G)]^{1/p}$, (iv) and (v) hold, and (vi) and (vii) are then seen to hold.

5.2 Relations among (α, p)-capacities

First we compute the (α, p)-capacity of balls. For this purpose, define

$$h_{\alpha,p}(r) = \begin{cases} r^{n-\alpha p} & \alpha p < n, \\ \{\log(2R/r)\}^{1-p} & \alpha p = n, \end{cases}$$

where $0 < r \leqq R$; if $r \geqq R$, then we define $h_{\alpha,p}(r) = h_{\alpha,p}(R)$.

THEOREM 2.1. *Suppose $\alpha p \leqq n$. If $B(a, r) \subseteqq B(0, R)$, then*

$$M^{-1} h_{\alpha,p}(r) \leqq C_{\alpha,p}(B(a, r); B(0, 2R)) \leqq M h_{\alpha,p}(r),$$

where M is a positive constant independent of $B(a, r)$ and R.

PROOF. Let $B(a, r) \subseteqq B(0, R)$, and consider the function

$$f_r(y) = |a - y|^{-\alpha} [|a - y|^{n-\alpha p}]^{-p'/p} \qquad \text{for } y \in B(0, 2R) - B(x, r);$$

set $f_r = 0$ outside $B(0, 2R)$. For simplicity, set $h(r) = h_{\alpha,p}(r)$. If $x \in B(a, r)$, then $|x - y| \leqq |x - a| + |a - y| \leqq 2|a - y|$ for $y \in \mathbf{R}^n - B(a, r)$, so that

$$\begin{aligned} \int |x - y|^{\alpha - n} f_r(y)\, dy &\geqq 2^{\alpha - n} \int_{B(0,2R) - B(a,r)} |a - y|^{-n} [|a - y|^{n-\alpha p}]^{-p'/p}\, dy \\ &\geqq M [h(r)]^{-p'/p}. \end{aligned}$$

Hence it follows that

$$\begin{aligned} C_{\alpha,p}(B(a,r);B(0,2R)) &\leqq \int [f_r(y)/M[h(r)]^{-p'/p}]^p \, dy \\ &\leqq M[h(r)]^{p'} \int [f_r(y)]^p \, dy \\ &\leqq M[h(r)]^{p'} \int |a-y|^{-\alpha p}[|a-y|^{n-\alpha p}]^{-p'} \, dy \\ &\leqq M[h(r)]^{p'+(1-p')} = Mh(r). \end{aligned}$$

Conversely, take a nonnegative measurable function f such that $f = 0$ outside $B(0,2R)$ and $U_\alpha f \geqq 1$ on $B(a,r)$. Since $B(a,r) \subseteqq B(0,R)$, we see that

$$\begin{aligned} \frac{1}{|B(a,r)|}\int_{B(a,r)} dx &\leqq \frac{1}{|B(a,r)|}\int_{B(a,r)} \left(\int |x-y|^{\alpha-n} f(y)dy\right) dx \\ &= \int \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} |x-y|^{\alpha-n}dx\right) f(y)dy \\ &\leqq M \int_{B(0,2R)} (r+|y|)^{\alpha-n} f(y)dy \\ &\leqq M[h_{\alpha,p}(r)]^{-1/p}\|f\|_p, \end{aligned}$$

which gives the left inequality.

LEMMA 2.1. *Let $1 < q < \infty$. If $U_\alpha\mu \in L^q$, then there exists a function k on $(0,\infty)$ such that*

(i) *k is positive, decreasing and continuous on $(0,\infty)$;*

(ii) $\int k(|x-y|) \, d\mu(y) \in L^q$;

(iii) $\lim_{r\to 0} k(r)/r^{\alpha-n} = \infty$.

The proof is similar to that of Theorem 7.4 in Chapter 2, and we leave it to the reader.

THEOREM 2.2. *Suppose $\alpha p \leqq n$. If $H_{h_{\alpha,p}}(E) < \infty$, then $C_{\alpha,p}(E) = 0$.*

PROOF. Suppose $H_{h_{\alpha,p}}(E) < \infty$ but $B_{\alpha,p}(E) > 0$. Here we may assume that E is compact, and then find a unit measure μ supported by $\overline{E}$ for which $U_\alpha\mu \in L^{p'}$. Let k be a function as in Lemma 2.1. Since, by assumption, $c_{\alpha,p}(E \cap B(a,1); B(a,2)) > 0$ for some a, we may assume, without loss of generality, that

$$c_{\alpha,p}(E \cap \mathbf{B}; B(0,2)) > 0. \tag{2.1}$$

Let $0 < \varepsilon < 1/2$ be given, and set

$$\delta(\varepsilon) = \inf_{r<\varepsilon} \ k(r)/r^{\alpha-n}.$$

Then it is easy to see that

$$C_{k,p}(B(a,r);B(0,2)) \leqq C_{k,p}(B(a,r);B(a,2\varepsilon)) \leqq [\delta(2\varepsilon)]^p C_{\alpha,p}(B(a,r);B(a,2\varepsilon))$$

whenever $B(a,r) \subseteqq \mathbf{B}$. Now find a covering $\{B(a_j,r_j)\}$ of $E \cap \mathbf{B}$ such that $10r_j < \varepsilon$ and

$$\sum_j h_{\alpha,p}(r_j) \leqq H_{h_{\alpha,p}}(E \cap \mathbf{B}) + \varepsilon.$$

By a covering lemma (see Theorem 10.1 in Chapter 1), we can choose a disjoint subfamily $\{B(a_{j'},r_{j'})\}$ for which $\{B(a_{j'},5r_{j'})\}$ covers $E \cap \mathbf{B}$. Then we have

$$\begin{aligned} C_{k,p}(E\cap \mathbf{B};B(0,2)) &\leqq \sum_{j'} C_{k,p}(B(a_{j'},5r_{j'});B(a_{j'},2\varepsilon)) \\ &\leqq [\delta(2\varepsilon)]^p \sum_{j'} C_{\alpha,p}(B(a_{j'},5r_{j'});B(a_{j'},2\varepsilon)) \\ &\leqq M[\delta(2\varepsilon)]^p \sum_{j'} h_{\alpha,p}(r_{j'}) \\ &\leqq M[\delta(2\varepsilon)]^p \{H_{h_{\alpha,p}}(E\cap \mathbf{B}) + \varepsilon\}. \end{aligned}$$

Hence it follows that $C_{k,p}(E \cap \mathbf{B}; B(0,2)) = 0$, which contradicts (2.1).

THEOREM 2.3. *Suppose $\alpha p \leqq n$. If $C_{\alpha,p}(E) = 0$, then $H_\gamma(E) = 0$ for $\gamma > n - \alpha p$.*

PROOF. Let $\gamma > n - \alpha p$ and suppose $H_\gamma(E) > 0$. We may assume that E is compact. By a theorem of Frostman used in the proof of Theorem 7.6 in Chapter 2, we can find a positive measure μ supported by E such that

$$\mu(B(x,r)) \leqq r^\gamma \qquad \text{for any ball } B(x,r).$$

Write

$$\begin{aligned} U_\alpha\mu(x) &= \int r^{\alpha-n}\, d\mu(B(x,r)) \\ &\leqq M\int_0^1 \mu(B(x,r))r^{\alpha-n-1}\, dr + M \\ &\leqq M\left(\sup_{r<1}\, \mu(B(x,r)r^{-\gamma}\right)^{1/p} \int_0^1 [\mu(B(x,r))]^{1/p'} r^{\gamma/p+\alpha-n-1}\, dr + M \\ &\leqq M\int_0^1 [\mu(B(x,r))]^{1/p'} r^{\gamma/p+\alpha-n-1}\, dr + M. \end{aligned}$$

Hence, for any bounded open set G, we have by Minkowski's inequality for integral

$$\begin{aligned} \left(\int_G [U_\alpha\mu(x)]^{p'}\, dx\right)^{1/p'} &\leqq M\int_0^1 \left(\int_G \mu(B(x,r))\, dx\right)^{1/p'} r^{\gamma/p+\alpha-n-1}\, dr + M \\ &\leqq M[\mu(E)]^{1/p'} \int_0^1 r^{n/p'} r^{\gamma/p+\alpha-n-1}\, dr + M < \infty, \end{aligned}$$

since $n/p' + \gamma/p + \alpha - n = [\gamma - (n - \alpha p)]/p > 0$. Thus it follows that $c_{\alpha,p}(E; G) > 0$ if $E \subseteqq G$, which gives a contradiction.

COROLLARY 2.1. *Suppose $\alpha p \leqq n$. If $C_{\alpha,p}(E) = 0$, then E has Hausdorff dimension at most $n - \alpha p$.*

THEOREM 2.4. *Let $0 < \alpha, \beta < n$ and $1 < p, q < \infty$. If $\alpha p < \beta q$, then $C_{\beta,q}(E) = 0$ implies $C_{\alpha,p}(E) = 0$.*

PROOF. Suppose $C_{\beta,q}(E) = 0$. If $\alpha p < \beta q$, then Theorem 2.3 implies that $H_\gamma(E) = 0$ for any $\gamma > n - \beta q$. Since $n - \alpha p > n - \beta q$, we have $H_{n-\alpha p}(E) = 0$. Now Theorem 2.2 implies that $C_{\alpha,p}(E) = 0$.

5.3 Continuity properties

In view of Sobolev's theorem, if $\alpha p > n$, then $U_\alpha f$ is continuous on $\mathbf{R}^n$ whenever $f \in L^p(\mathbf{R}^n)$ and $U_\alpha |f| \not\equiv \infty$. In case $\alpha p \leqq n$, $U_\alpha f$ may not be continuous on $\mathbf{R}^n$ for the above f. This is closely related to the fact that $C_{\alpha,p}(\{a\}) = 0$ for a point a if and only if $\alpha p \leqq n$.

Throughout this section, assume that $\alpha p = n$. Let φ be a positive nondecreasing function on the interval $(0, \infty)$ satisfying

$$A^{-1}\varphi(r) \leqq \varphi(r^2) \leqq A\varphi(r). \tag{φ}$$

Our first aim in this section is to discuss the continuity of $U_\alpha f$ when

$$\int (1 + |y|)^{\alpha - n} |f(y)| \, dy < \infty \tag{3.1}$$

and

$$\int \Phi_p(|f(y)|) \, dy < \infty, \tag{3.2}$$

where $\Phi_p(r) = r^p \varphi(r)$.

For the sake of convenience, set

$$\varphi(0) = \lim_{r \to 0} \varphi(r) \geqq 0. \tag{3.3}$$

By condition (φ), we have the doubling condition

$$A^{-1}\varphi(r) \leqq \varphi(2r) \leqq A\varphi(r) \tag{$\varphi1$}$$

and for $\gamma > 1$,

$$A_\gamma^{-1}\varphi(r) \leqq \varphi(r^\gamma) \leqq A_\gamma \varphi(r). \tag{$\varphi2$}$$

LEMMA 3.1. *If $\gamma > 0$, then*

$$(\varphi 3) \qquad s^\gamma \varphi(s^{-1}) \leqq M t^\gamma \varphi(t^{-1}) \qquad \text{whenever } t > s > 0.$$

PROOF. Let $0 < s < t \leqq A^{-1/\gamma}$. If m is the positive integer for which $(t^{-1})^{2^m} \leqq s^{-1} < (t^{-1})^{2^{m+1}}$, then we have by condition (γ)

$$\begin{aligned} \varphi(s^{-1}) &\leqq \varphi((t^{-1})^{2^{m+1}}) \leqq A^{m+1}\varphi(t^{-1}) \\ &\leqq A^{2^m-1}[A\varphi(t^{-1})] \leqq [t^{-\gamma}]^{2^m-1}[A\varphi(t^{-1})] \leqq s^{-\gamma}t^\gamma[A\varphi(t^{-1})], \end{aligned}$$

so that $(\varphi 3)$ holds for $0 < s < t \leqq A^{-1/\gamma}$. Since φ is positive and nondecreasing, we see that

$$(3.4) \qquad s^\gamma \varphi(s^{-1}) \leqq M t^\gamma \varphi(t^{-1}) \qquad \text{whenever } 0 < s < t \leqq 1.$$

If we consider $\psi(r) = [\varphi(r^{-1})]^{-1}$, then ψ satisfies the same conditions as φ, so that

$$(3.5) \qquad \frac{s^\gamma}{\varphi(s)} \leqq M \frac{t^\gamma}{\varphi(t)} \qquad \text{whenever } 0 < s < t \leqq 1.$$

In particular, if we set $t = 1$ in (3.5), then

$$(3.6) \qquad M^{-1}s^\gamma \leqq \varphi(s) \qquad \text{whenever } 0 < s < 1.$$

If we apply (3.6) for γ' smaller than γ, then we see that

$$\liminf_{t\to\infty}\ t^\gamma\varphi(t^{-1}) \geqq \limsup_{t\to\infty}\ M^{-1}t^{\gamma-\gamma'} = \infty,$$

so that (3.4) holds for $0 < s < t$ and $s < 1$. If $t > s \geqq 1$, then $0 < t^{-1} < s^{-1} \leqq 1$ and hence (3.5) gives

$$\frac{t^{-\gamma}}{\varphi(t^{-1})} \leqq M \frac{s^{-\gamma}}{\varphi(s^{-1})},$$

which implies that (3.4) holds in this case, too.

In view of (3.5), it follows also that

$$(\varphi 4) \qquad \frac{s^\gamma}{\varphi(s)} \leqq M \frac{t^\gamma}{\varphi(t)} \qquad \text{whenever } t > s > 0.$$

THEOREM 3.1. *Let φ satisfy the following condition :*

$$(3.7) \qquad \int_0^1 [\varphi(r^{-1})]^{-1/(p-1)} r^{-1} dr < \infty,$$

and set

$$\varphi^*(r) = \left(\int_0^r \ [\varphi(t^{-1})]^{-1/(p-1)} t^{-1} dt \right)^{1-1/p}.$$

If f satisfies (3.1) *and* (3.2)*, then $U_\alpha f$ is continuous on $\mathbf{R}^n$ and, moreover,*

$$|U_\alpha f(x) - U_\alpha f(z)| = o(\varphi^*(|x-z|)) \qquad \text{as } |x-z| \to 0.$$

PROOF. Let $r = |x-z| < 1/2$ and write

$$\begin{aligned} U_\alpha f(z) &= \int_{B(x,2r)} \ |z-y|^{\alpha-n} f(y) dy + \int_{\mathbf{R}^n - B(x,2r)} \ |z-y|^{\alpha-n} f(y) dy \\ &= u_1(z) + u_2(z). \end{aligned}$$

For $0 < \gamma < \alpha$, we have by Hölder's inequality

$$\begin{aligned} |u_1(z)| &\leqq \int_{B(z,3r)} \ |z-y|^{\alpha-n-\gamma} dy \\ &\quad + \int_{\{x:B(z,3r);|f(y)|>|z-y|^{-\gamma}\}} [|z-y|^{\alpha-n} \varphi(|z-y|^{-\gamma})^{-1/p}] \ [|f(y)|\varphi(|f(y)|)^{1/p}] dy \\ &\leqq Mr^{\alpha-\gamma} + \left(\int_{B(z,3r)} \ [|z-y|^{\alpha-n}\varphi(|z-y|^{-\gamma})^{-1/p}]^{p'} dy \right)^{1/p'} \\ &\qquad\qquad \times \left(\int_{B(z,3r)} \ [|f(y)|\varphi(|f(y)|)^{1/p}]^p dy \right)^{1/p} \\ &= Mr^{\alpha-\gamma} + M \left(\int_0^{3r} \ [\varphi(t^{-\gamma})]^{-p'/p} t^{-1} dt \right)^{1/p'} \left(\int_{B(z,3r)} \ \Phi_p(|f(y)|) dy \right)^{1/p}, \end{aligned}$$

so that we have by ($\varphi 2$)

$$|u_1(z)| \leqq Mr^{\alpha-\gamma} + M\varphi^*(r) \left(\int_{B(x,4r)} \ \Phi_p(|f(y)|) dy \right)^{1/p}.$$

On the other hand, we note that

$$||x-y|^{\alpha-n} - |z-y|^{\alpha-n}| \leqq Mr|x-y|^{\alpha-n-1} \qquad \text{whenever } y \in \mathbf{R}^n - B(x,2r),$$

so that

$$\begin{aligned} &\int_{\mathbf{R}^n - B(x,2r)} \ ||x-y|^{\alpha-n} - |z-y|^{\alpha-n}||f(y)| \ dy \\ \leqq \ & Mr \int_{\mathbf{R}^n - B(x,2r)} \ |x-y|^{\alpha-n-1} |f(y)| \ dy. \end{aligned}$$

Hence for $\alpha - 1 < \gamma < \alpha$, we have as above

$$\begin{aligned}|u_2(x) - u_2(z)| &\leqq Mr \int_{\mathbf{R}^n - B(x,2r)} |x-y|^{\alpha-n-1}|f(y)|\, dy \\ &\leqq Mr \int_{\mathbf{R}^n - B(x,2r)} |x-y|^{\alpha-n-1-\gamma}\, dy + Mr\varphi(r^{-\gamma})^{-1/p} \\ &\quad \times \int_{\{x:\mathbf{R}^n - B(x,2r); |f(y)|>r^{-\gamma}\}} |x-y|^{\alpha-n-1}[|f(y)|\varphi(|f(y)|)^{1/p}]dy \\ &\leqq Mr^{\alpha-\gamma} + Mr[\varphi(r^{-\gamma})]^{-1/p} \left(\int_{\mathbf{R}^n - B(x,2r)} |x-y|^{(\alpha-n-1)p'} dy \right)^{1/p'} \\ &\qquad \times \left(\int_{\mathbf{R}^n - B(x,2r)} \Phi_p(|f(y)|)dy \right)^{1/p} \\ &\leqq Mr^{\alpha-\gamma} + M\varphi(r^{-\gamma})^{-1/p} \left(\int \Phi_p(|f(y)|)dy \right)^{1/p}.\end{aligned}$$

By (φ), we see that

$$\varphi^*(r) \geqq \left(\int_{r^2}^{r} [\varphi(t^{-1})]^{-1/(p-1)} t^{-1} dt \right)^{1/p'} \geqq M[\varphi(r^{-1})]^{-1/p}[\log(1/r)]^{1/p'}.$$

Further, by an application of $(\varphi 4)$ with $[\varphi(r^{-1})]^{-1}$,

$$Ms^{\alpha-\gamma} \leqq [\varphi(s^{-1})]^{-1} \qquad \text{whenever } 0 < s < 1. \tag{3.8}$$

Thus we establish

$$|u_2(x) - u_2(z)| \leqq M\varphi^*(r)[\log(1/r)]^{-1/p'} \left(\int \Phi_p(|f(y)|)dy \right)^{1/p}.$$

Now it follows that

$$\begin{aligned}|U_\alpha f(x) - U_\alpha f(z)| &\leqq Mr^{\alpha-\gamma} + M\varphi^*(r) \left(\int_{B(x,4r)} \Phi_p(|f(y)|)dy \right)^{1/p} \\ &\quad + M\varphi^*(r)[\log(1/r)]^{-1/p'} \left(\int \Phi_p(|f(y)|)dy \right)^{1/p},\end{aligned}$$

which together with (3.8) proves the required result.

Remark 3.1. Consider the function

$$\varphi(r) = [\log(1+r)]^\delta.$$

Then φ satisfies (3.7) if and only if $\delta > p - 1$. The same is true for

$$\varphi(r) = [\log(1+r)]^{p-1}[\log(1+\log(1+r))]^\delta,$$

and so on.

REMARK 3.2. If $\int_0^1 [\varphi(r^{-1})]^{-1/(p-1)} r^{-1} dr = \infty$, then we can find a nonnegative function f satisfying (3.2) for which $U_\alpha f(0) = \infty$.

In fact, let

$$\psi(r) = \int_r^1 [\varphi(t^{-1})]^{-1/(p-1)} t^{-1} dt$$

and consider the function

$$f(y) = [\psi(|y|)]^{-1/p} [\log \psi(|y|)]^{-\delta} |y|^{-\alpha} [\varphi(|y|^{-1})]^{-p'/p}$$

for $y \in B(0, r_0)$, where $1/p < \delta < 1$. Then by change of variables $t = \psi(r)$, we have

$$\int_{B(0,r_0)} |y|^{\alpha-n} f(y) dy = M \int_{t_0}^{\infty} t^{-1/p} [\log t]^{-\delta} dt = \infty,$$

where $t_0 = \psi(r_0)$. Since $f(y) \leqq M|y|^{-\alpha}$,

$$\begin{aligned} \Phi_p(f(y)) &\leqq \left\{ t^{-1/p} (\log t)^{-\delta} |y|^{-\alpha} [\varphi(|y|^{-1})]^{-p'/p} \right\}^p \varphi(M|y|^{-\alpha}) \\ &\leqq M t^{-1} (\log t)^{-\delta p} [\varphi(|y|^{-1})]^{-p'+1} |y|^{-n}, \end{aligned}$$

so that, in the same way as above, we have

$$\int_{B(0,r_0)} \Phi_p(f(y)) dy \leqq M \int_{t_0}^{\infty} t^{-1} (\log t)^{-\delta p} dt < \infty.$$

5.4 Fine limits

In this section, let $\alpha p \leqq n$. In this case, $U_\alpha f$ may not be continuous by Remark 3.1, when $f \in L^p$. We are concerned with weak limits, which are an extension of fine limits of potentials of measures.

First we begin with maximum principle for nonlinear potentials.

THEOREM 4.1 (maximum principle for nonlinear potentials). *Let G be an open set in $\mathbf{R}^n$. If*

$$\int_G |x-y|^{\alpha-n} [U_\alpha \mu(y)]^{1/(p-1)} dy \leqq 1$$

for any $x \in S_\mu$, then

$$\int_G |x-y|^{\alpha-n} [U_\alpha \mu(y)]^{1/(p-1)} dy \leqq M$$

for every $x \in \mathbf{R}^n$, where $M = (n/\alpha) 2^n 3^{p'(n-\alpha)}$.

PROOF. Let $a \in \mathbf{R}^n - S_\mu$ and take $a^* \in S_\mu$ such that $2r \equiv |a - a^*| = \mathrm{dist}(a, S_\mu)$. Set

$$\begin{aligned} u_1(x) &= \int_{B(a,r)} |x-y|^{\alpha-n}[U_\alpha\mu(y)]^{1/(p-1)}dy, \\ u_2(x) &= \int_{G-B(a,r)} |x-y|^{\alpha-n}[U_\alpha\mu(y)]^{1/(p-1)}dy. \end{aligned}$$

If $y \in \mathbf{R}^n - B(a,r)$, then $|a^* - y| \leqq |a^* - a| + |a - y| \leqq 3|a-y|$, so that

$$u_2(a) \leqq 3^{n-\alpha} u_2(a^*).$$

On the other hand, if $z \in S_\mu$ and $y \in B(a,r)$, then $|z-y| \geqq |a-z| - |a-y| \geqq |a-z|/2$, so that

$$\begin{aligned} u_1(a) &\leqq [2^{n-\alpha}U_\alpha\mu(a)]^{1/(p-1)} \int_{B(a,r)} |a-y|^{\alpha-n}dy \\ &\leqq [2^{n-\alpha}U_\alpha\mu(a)]^{1/(p-1)}(\omega_n/\alpha)r^\alpha \\ &\leqq [2^{n-\alpha}U_\alpha\mu(a)]^{1/(p-1)}(n/\alpha)2^n 3^{n-\alpha} \int_{B(a,r)} |a^*-y|^{\alpha-n}dy. \end{aligned}$$

Since $|z-y| \leqq |a-z| + |a-y| \leqq (3/2)|a-z|$ when $z \in S_\mu$ and $y \in B(a,r)$, it follows that

$$u_1(a) \leqq (n/\alpha)2^n 3^{(n-\alpha)p'} u_1(a^*).$$

Thus Theorem 4.1 is proved.

LEMMA 4.1. *If μ is a measure with compact support, then*

$$\int_{B(0,2R)} |x-y|^{\alpha-n}[U_\alpha\mu(y)]^{1/(p-1)}dy \geqq M \int_0^R \left(r^{\alpha p-n}\mu(B(x,r))\right)^{1/(p-1)} r^{-1}dr$$

for every $x \in B(0,R)$.

PROOF. For $y \in B(x,r)$, we have

$$U_\alpha\mu(y) \geqq (2r)^{\alpha-n}\mu(B(y,2r)) \geqq (2r)^{\alpha-n}\mu(B(x,r)),$$

so that

$$\begin{aligned} &\int_{B(0,2R)} |x-y|^{\alpha-n}[U_\alpha\mu(y)]^{1/(p-1)}dy \\ \geqq\ &\int_0^R |B(x,r)| \left((2r)^{\alpha-n}\mu(B(x,r))\right)^{1/(p-1)} d(-r^{\alpha-n}) \\ \geqq\ &M \int_0^R \left(r^{\alpha p-n}\mu(B(x,r))\right)^{1/(p-1)} r^{-1}dr. \end{aligned}$$

THEOREM 4.2. *Let $f \in L^p(\mathbf{R}^n)$ and set*

$$J_{\alpha,p}(x) = \int_0^1 \left(r^{\alpha p - n} \int_{B(x,r)} |f(y)|^p dy \right)^{1/(p-1)} \frac{dr}{r}.$$

Then $C_{\alpha,p}(\{x : J_{\alpha,p}(x) = \infty\}) = 0$.

PROOF. Suppose $C_{\alpha,p}(K) > 0$ for some compact set K such that $K \subseteqq \{x : J_{\alpha,p}(x) = \infty\}$. Then, in view of Theorems 1.6 and 4.1, we can find a positive measure μ supported by K for which

$$\int_G |x-y|^{\alpha - n} [U_\alpha \mu(y)]^{1/(p-1)} dy \leqq M,$$

where $K \subseteqq G$. Let $d\nu(y) = f(y)^p dy$ and write, by Theorems 10.4 and 10.5 in Chapter 1,

$$\nu = g\mu + \sigma,$$

where σ is singular and

$$g(x) = \lim_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \qquad \text{for } \mu\text{-a.e. } x.$$

In view of Lemma 4.1, we see that $J_{\alpha,p}(x) < \infty$ for μ-a.e. x. But, since $\mu(K) > 0$, a contradiction follows.

A set E is called (α, p)-thin at x_0 if

$$\sum_{j=1}^{\infty} \left(2^{j(n-\alpha p)} C_{\alpha,p}(E_j; B_j) \right)^{1/(p-1)} < \infty,$$

where $E_j = \{x \in E : 2^{-j} \leqq |x - x_0| < 2^{-j+1}\}$ and $B_j = \{x : 2^{-j-1} < |x - x_0| < 2^{-j+2}\}$.

THEOREM 4.3. *Let f be a function in $L^p(\mathbf{R}^n)$ satisfying* (3.1). *If $J_{\alpha,p}(x_0) < \infty$ and $U_\alpha|f|(x_0) < \infty$, then there exists a set E for which E is (α, p)-thin at x_0 and*

$$\lim_{x \to x_0, x \in \mathbf{R}^n - E} U_\alpha f(x) = U_\alpha f(x_0). \tag{4.1}$$

If (4.1) holds, then we say that $U_\alpha f$ has an (α, p)-fine limit $U_\alpha f(x_0)$.

COROLLARY 4.1. *If f is as in Theorem 4.3, then $U_\alpha f$ has an (α, p)-fine limit (α, p)-q.e. on $\mathbf{R}^n$.*

PROOF OF THEOREM 4.3. For simplicity, assume that $x_0 = 0$ and $f \geqq 0$. For $x \neq 0$, write

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x,|x|/2)} |x-y|^{\alpha-n} f(y) dy + \int_{\mathbf{R}^n - B(x,|x|/2)} |x-y|^{\alpha-n} f(y) dy \\ &= u_1(x) + u_2(x). \end{aligned}$$

If $y \in \mathbf{R}^n - B(x, |x|/2)$, then $|y| \leqq |x| + |x-y| \leqq 3|x-y|$, so that

$$|x-y|^{\alpha-n} f(y) \leqq 3^{n-\alpha} |y|^{\alpha-n} f(y).$$

Since $U_\alpha f(0) < \infty$ by assumption, we can apply Lebesgue's dominated convergence theorem to see that

$$\lim_{x \to 0} u_2(x) = U_\alpha f(0). \tag{4.2}$$

To deal with u_1, in view of $J_{\alpha,p}(x_0) < \infty$, we take a sequence $\{a_j\}$ such that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j \left(2^{j(n-\alpha p)} \int_{B_j} f(y)^p dy \right)^{1/(p-1)} < \infty. \tag{4.3}$$

Now consider the sets

$$E_j = \{x : 2^{-j} \leqq |x| < 2^{-j+1}, u_1(x) > a_j^{-1/p'}\}.$$

If $x \in E_j$ and $y \in B(x, |x|/2)$, then

$$u_1(x) \leqq \int_{B_j} |x-y|^{\alpha-n} f(y) dy,$$

so that

$$C_{\alpha,p}(E_j; B_j) \leqq a_j^{p/p'} \int_{B_j} f(y)^p dy.$$

Hence it follows from (4.3) that

$$\sum_{j=1}^{\infty} \left(2^{j(n-\alpha p)} C_{\alpha,p}(E_j; B_j) \right)^{1/(p-1)} < \infty.$$

This implies that $E = \bigcup_{j=1}^{\infty} E_j$ is (α, p)-thin at 0. On the other hand,

$$\limsup_{x\to 0, x \in \mathbf{R}^n - E} u_1(x) \leqq \limsup_{j\to\infty} a_j^{-1/p'} = 0.$$

This, together with (4.2), implies (4.1), and the proof is completed.

We have another type of fine limit result, which is better in studying the existence of radial limits.

THEOREM 4.4. *Let f be a function in $L^p(\mathbf{R}^n)$ satisfying* (3.1). *If*

$$\int_{B(x_0,1)} |x_0 - y|^{\alpha p - n} |f(y)|^p dy < \infty, \tag{4.4}$$

then there exists a set E for which

$$\lim_{x\to x_0, x\in\mathbf{R}^n-E} U_\alpha f(x) = U_\alpha f(x_0)$$

and

$$\sum_{j=1}^{\infty} 2^{j(n-\alpha p)} C_{\alpha,p}(E_j; B_j) < \infty. \tag{4.5}$$

THEOREM 4.5. *Let f be a function in $L^p(\mathbf{R}^n)$ satisfying* (3.1). *Then there exists a set E satisfying* (4.5) *for which*

$$\lim_{x\to x_0, x\in\mathbf{R}^n-E} [\kappa(|x|)]^{-1} U_\alpha f(x) = 0.$$

PROOF. For simplicity, assume that $x_0 = 0$, and $f \geqq 0$, as before. Let $r = |x|/2 > 0$ and $\varepsilon > 0$, write

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x,r)} |x-y|^{\alpha-n} f(y)dy + \int_{B(0,4r)-B(x,r)} |x-y|^{\alpha-n} f(y)dy \\ &\quad + \int_{B(0,\varepsilon)-B(0,4r)} |x-y|^{\alpha-n} f(y)dy + \int_{\mathbf{R}^n-B(0,\varepsilon)} |x-y|^{\alpha-n} f(y)dy \\ &= u_1(x) + u_2(x) + u_3(x) + u_4(x). \end{aligned}$$

By Hölder's inequality we have

$$\begin{aligned} u_2(x) &\leqq r^{\alpha-n} \int_{B(0,4r)} f(y)dy \\ &\leqq r^{\alpha-n} |B(0,4r)|^{1/p'} \left(\int_{B(0,4r)} f(y)^p dy \right)^{1/p} \\ &\leqq M r^{(\alpha p-n)/p} \left(\int_{B(0,4r)} f(y)^p dy \right)^{1/p} \end{aligned}$$

and

$$\begin{aligned} u_3(x) &= \int_{B(0,\varepsilon)-B(0,4r)} (|y|/2)^{\alpha-n} f(y)dy \\ &\leqq 2^{n-\alpha} \left(\int_{B(0,\varepsilon)-B(0,4r)} |y|^{p'(\alpha-n)} dy \right)^{1/p'} \left(\int_{B(0,\varepsilon)-B(0,4r)} f(y)^p dy \right)^{1/p} \\ &\leqq M\kappa(r) \left(\int_{B(0,\varepsilon)} f(y)^p dy \right)^{1/p}. \end{aligned}$$

Hence it follows that

$$\limsup_{x\to 0} [\kappa(|x|)]^{-1} [u_2(x) + u_3(x) + u_4(x)] \leqq M \left(\int_{B(0,\varepsilon)} f(y)^p dy \right)^{1/p},$$

which implies that the left hand-side is equal to zero by the arbitrariness of ε. Take a sequence $\{a_j\}$ such that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j \int_{B_j} f(y)^p dy < \infty.$$

As in the proof of Theorem 4.3, consider the sets

$$E_j = \{x : 2^{-j} \leqq |x| < 2^{-j+1}, u_1(x) > a_j^{-1/p}\kappa(2^{-j})\}$$

and

$$E = \bigcup_{j=1}^{\infty} E_j.$$

Then it is easy to see that

$$[\kappa(2^{-j})]^{-p} C_{\alpha,p}(E_j; B_j) \leqq a_j \int_{B_j} f(y)^p dy,$$

so that E satisfies (4.5). Moreover,

$$\limsup_{x\to 0, x\in \mathbf{R}^n - E} [\kappa(|x|)]^{-1} u_1(x) \leqq \limsup_{j\to\infty} a_j^{-1/p} = 0.$$

Thus Theorem 4.5 is obtained.

5.5 Contractive property of (α, p)-capacities

First we deduce a contractive property of (α, p)-capacity, which will be used for the study of radial limit result.

Now, let f be a measurable function on $\mathbf{R}^n$. For $x = (x_1, x_2, ..., x_n) = (x_1, x')$, define the symmetrization with respect to the hyperplane $\{x : x_1 = 0\}$ by setting

$$f^*(x) = f(\cdot, x')^*(x_1).$$

If f is a characteristic function of a rectangle I with sides parallel to the coordinate axes, then we see easily that f^* is also a characteristic function of a rectangle I^* which is obtained from I by translation. Hence, if $f = \sum_{j=1}^{m} a_j \chi_{I_j}$, where $0 < a_1 < a_2 < \cdots < a_m$ and $\{I_j\}$ are mutually disjoint rectangles with sides parallel to the coordinate axes, then it follows that f^* is measurable. By a passage of limit process with the aid of Lemma 1.3 in Chapter 4, we deduce the measurability of symmetrizations.

Lemma 5.1. *If f is bounded and measurable on $\mathbf{R}^n$, then f^* is also bounded and measurable on $\mathbf{R}^n$.*

LEMMA 5.2. *If $f \in L^p(\mathbf{R}^n)$, then*

$$\int |f(x)|^p dx = \int f^*(x)^p dx.$$

LEMMA 5.3. *If k is nonnegative and nonincreasing on $(0, \infty)$, then*

$$\int k(|x-y|)|f(y)|dy \leqq \int k(|y|)f^*(y)dy.$$

For $t > 0$, write

$$R(t) = \{x = (x_1, ..., x_n) : |x_j| < t \quad (j = 1, 2, ..., n)\}.$$

For a set E, denote by E^* the projection of E to the hyperplane $\mathbf{H}$.

THEOREM 5.1. *For $E \subseteqq R(t)$,*

$$C_{\alpha,p}(E^*; R(t)) \leqq C_{\alpha,p}(E; R(t)).$$

PROOF. Let f be a competing function for $C_{\alpha,p}(E; R(t))$. Then Lemma 5.3 implies that

$$\int |x^* - y|^{\alpha-n} f^*(y)dy \geqq \int |x-y|^{\alpha-n} f(y)dy \geqq 1$$

for all $x^* = (0, x') \in E^*$, where $x = (x_1, x') \in E$. Hence it follows from Lemma 5.2 that

$$C_{\alpha,p}(E^*; R(t)) \leqq \|f^*\|_p^p = \|f\|_p^p,$$

which proves the required inequality.

Let G be a bounded open set. A mapping T: $G \to TG$ is said to be Lipschitzian if there exists $A > 0$ such that

$$A^{-1}|x-y| \leqq |Tx - Ty| \leqq A|x-y| \qquad \text{whenever } x,\, y \in G.$$

LEMMA 5.4. *If G and T are as above, then*

$$M^{-1}C_{\alpha,p}(E; G) \leqq C_{\alpha,p}(TE; TG) \leqq MC_{\alpha,p}(E; G)$$

for any set $E \subseteqq G$, where $M = A^{n+p(2n-\alpha)}$.

PROOF. Let f be a competing function for $C_{\alpha,p}(E; G)$, and consider

$$g(z) = f(T^{-1}z) \qquad \text{for } z \in TG.$$

Then, if $x \in E$, then

$$\int |Tx - z|^{\alpha-n} g(z)dz \geqq M^{\alpha-2n} \int |x-y|^{\alpha-n} f(y)dy,$$

so that

$$C_{\alpha,p}(TE;TG) \leqq M^{p(2n-\alpha)} \|g\|_p^p \leqq M^{n+p(2n-\alpha)} \|f\|_p^p.$$

Thus the required inequality follows.

COROLLARY 5.1. *For $r > 0$,*

$$C_{\alpha,p}(rE; B(0,rN)) = r^{n-\alpha p} C_{\alpha,p}(E; B(0,N)),$$

where $rE = \{rx : x \in E\}$.

LEMMA 5.5. *Let G and G' be bounded open sets in $\mathbf{R}^n$, and let F be a compact set in $G \cap G'$. Then*

$$C_{\alpha,p}(E;G') \leqq M C_{\alpha,p}(E;G)$$

whenever $E \subseteqq F$.

PROOF. First note that $\eta_0 = C_{\alpha,p}(F; G \cap G') < \infty$. To show this fact, assuming that $|F| > 0$, we have only to see that the potential $\int_F |x-y|^{\alpha-n}\, dy$ is bounded on $\mathbf{R}^n$. Let a =dist$(F, G \cap G')$ and take a competing function f for $C_{\alpha,p}(E;G)$. Setting $\eta = C_{\alpha,p}(E;G)$, we have for $x \in F$

$$\int_{G-B(x,a)} |x-y|^{\alpha-n} f(y)dy \leqq a^{\alpha-n} |G|^{1/p'} \|f\|_p \leqq M\eta.$$

Hence, if $M\eta < 1/2$, then

$$\int_{B(x,a)} |x-y|^{\alpha-n} f(y)dy > 1/2 \qquad \text{for any } x \in E.$$

Since $B(x,a) \subseteqq G'$ for $x \in E$, it follows that

$$C_{\alpha,p}(E;G') \leqq 2^p \int_{G'} f(y)^p dy,$$

so that

$$C_{\alpha,p}(E;G') \leqq 2^n C_{\alpha,p}(E;G)$$

whenever $\eta < 1/2M$. If $\eta \geqq 1/2M$, then

$$C_{\alpha,p}(E;G') \leqq \eta_0 \leqq \eta_0(2M\eta) \leqq 2M\eta_0 C_{\alpha,p}(E;G).$$

Thus the lemma is obtained.

THEOREM 5.2. *For $E \subseteqq B(0,2) - \mathbf{B}$, denote by $\tilde{E}$ the radial projection of E to the unit sphere $\mathbf{S}$, that is,*

$$\tilde{E} = \{\xi \in \mathbf{S} : r\xi \in E \quad \text{for some } r > 0\}.$$

Then

$$C_{\alpha,p}(\tilde{E}; B(0,3)) \leqq M C_{\alpha,p}(E; B(0,3)).$$

PROOF. Consider the truncated cones

$$\begin{aligned} F &= \{x = (x_1, x') : |x'| < x_1/2, 1 \leqq |x| \leqq 2\}, \\ G &= \{x = (x_1, x') : |x'| < x_1, 1/2 < |x| < 3\}. \end{aligned}$$

By the subadditivity of $C_{\alpha,p}$ and Lemma 5.5, it suffices to show that

$$C_{\alpha,p}(\tilde{E} \cap F; G) \leqq M C_{\alpha,p}(E \cap F; G) \tag{5.1}$$

for $E \subseteqq B(0,2) - \mathbf{B}$. Define the mapping T on G by setting

$$Tx = \left(|x|, \frac{x'}{|x|}\right);$$

note that $G' = TG = \{y = (y_1, y') : 1/2 < y_1 < 3, |y'| < 1/\sqrt{2}\}$. Since T is Lipschitzian, Lemma 5.4 implies that

$$C_{\alpha,p}(T(E \cap F); G') \leqq M C_{\alpha,p}(E \cap F; G).$$

In view of Theorem 5.1, we have

$$C_{\alpha,p}(T(E \cap F)^*; C) \leqq M C_{\alpha,p}(T(E \cap F); C),$$

where $C = \{x = (x_1, x') : -1 < x_1 < 3, |x'| < 1/\sqrt{2}\}$. Since $T(E \cap F)^* = T(\tilde{E} \cap F)$, (5.1) holds, and the required assertion is proved.

5.6 Radial limits

As an application of Theorem 4.4, we give the following radial limit result.

THEOREM 6.1. *Let f be a function in $L^p(\mathbf{R}^n)$ satisfying* (3.1) *and* (4.4). *Then there exists a set $E \subseteqq \mathbf{S}$ such that $C_{\alpha,p}(E) = 0$ and*

$$\lim_{r \to 0} U_\alpha f(x_0 + r\xi) = U_\alpha f(x_0) \qquad \text{for any } \xi \in \mathbf{S} - E.$$

To show this result, we need the following lemma.

Lemma 6.1. *If E satisfies (4.5), then $C_{\alpha,p}(E^*) = 0$, where*

$$E^* = \bigcap_{k=1}^{\infty} \left(\bigcup_{j=k}^{\infty} \tilde{E}_j \right).$$

Proof. By Corollary 5.1, we have

$$2^{j(n-\alpha p)} C_{\alpha,p}(E_j; B_j) = C_{\alpha,p}(2^j E_j; 2^j B_j),$$

where $2^j B_j \subseteqq B(0,4)$. In view of Theorem 5.2, we have

$$2^{j(n-\alpha p)} C_{\alpha,p}(E_j; B_j) \geqq M C_{\alpha,p}(\tilde{E}_j; B(0,4)).$$

Hence it follows that

$$C_{\alpha,p}(\bigcup_{j=k}^{\infty} \tilde{E}_j; B(0,4)) \leqq M^{-1} \sum_{j=k}^{\infty} 2^{j(n-\alpha p)} C_{\alpha,p}(E_j; B_j),$$

which tends to zero as $k \to \infty$, and we obtain the required result.

Noting that if $\xi \in \mathbf{S} - E^*$, then $r\xi \notin E$ for small $r > 0$, we have Theorem 6.1 as a consequence of Lemma 6.1 and Theorem 4.4.

The following is a consequence of Theorem 4.5.

Theorem 6.2. *Let f be a function in $L^p(\mathbf{R}^n)$ satisfying* (3.1). *Then there exists a set $E \subseteqq \mathbf{S}$ such that $C_{\alpha,p}(E) = 0$ and*

$$\lim_{r \to 0} [\kappa(r)]^{-1} U_\alpha f(x_0 + r\xi) = 0 \qquad \text{for any } \xi \in \mathbf{S} - E.$$

It is important to discuss the limits at infinity. The radial limit results will be obtained in the same way as Theorems 6.1 and 6.2, which will also be derived as direct consequences of Theorems 6.1 and 6.2 by considering the inversion. Here we discuss the limits along lines parallel to the coordinate axes.

Theorem 6.3. *Let $\alpha p < n$, and let f be a function in $L^p(\mathbf{R}^n)$. Then there exists a set $E' \subseteqq \mathbf{R}^{n-1}$ such that $C_{\alpha,p}(\{0\} \times E') = 0$ and*

$$\lim_{x_1 \to \infty} U_\alpha f(x_1, x') = 0 \qquad \text{for any } x' \in \mathbf{R}^{n-1} - E'.$$

Proof. We may assume that $f \geqq 0$. If $|x| > 2r > 0$, then we see that

$$\begin{aligned} u_r(x) &= \int_{B(0,r)} |x-y|^{\alpha-n} f(y)\,dy \\ &\leqq r^{\alpha-n} |B(0,r)|^{1/p'} \|f\|_p = M r^{\alpha-n/p} \|f\|_p. \end{aligned}$$

On the other hand, setting

$$v_r(x) = \int_{\mathbf{R}^n - B(0,r)} |x-y|^{\alpha-n} f(y)dy$$

and

$$A_{r,j} = \{x : v_r(x) > 1/(2j)\},$$

we have by the definition of $C_{\alpha,p}$,

$$C_{\alpha,p}(A_{r,j}) \leqq (2j)^p \int_{\mathbf{R}^n - B(0,r)} f(y)^p dy.$$

If r is sufficiently large, say $r \geqq r_j$, then

$$u_r(x) < 1/(2j) \qquad \text{whenever } |x| > 2r$$

and

$$C_{\alpha,p}(A_{r,j}) < 2^{-j}.$$

Consider the sets

$$B_j = \{x : |x| > 2r_j, U_\alpha f(x) > 1/j\}.$$

If $x \in B_j$, then $v_{r_j}(x) > 1/(2j)$, so that $B_j \subseteqq A_{r_j,j}$. Hence

$$C_{\alpha,p}(B_j) < 2^{-j}.$$

Now set

$$E_k = \bigcup_{j=k}^{\infty} B_j$$

and

$$E = \bigcap_{k=1}^{\infty} E_k^*,$$

where E_k^* denotes the projection of E_k to the hyperplane $\{0\} \times \mathbf{R}^{n-1}$. In view of Theorem 5.3, we have

$$C_{\alpha,p}(E_k^*) \leqq C_{\alpha,p}(E_k) \leqq \sum_{j=k}^{\infty} C_{\alpha,p}(B_j),$$

so that

$$C_{\alpha,p}(E) = 0.$$

If $(0,x') \notin E$, then $(0,x') \notin B_j$ for all $j \geqq k$. This implies that

$$U_\alpha(x_1,x') \leqq 1/j \qquad \text{whenever } x_1 > 2r_j \text{ and } j \geqq k,$$

so that

$$\lim_{x_1 \to \infty} U_\alpha f(x_1,x') = 0$$

as required.

5.7 Fine differentiability

For a point $x = (x_1, ..., x_n)$ and a multi-index $\lambda = (\lambda_1, ..., \lambda_n)$, we recall

$$x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n},$$

$$\lambda! = \lambda_1! \cdots \lambda_n!$$

and

$$D^\lambda = (\partial/\partial x_1)^{\lambda_1} \cdots (\partial/\partial x_n)^{\lambda_n}.$$

For a nonnegative integer m, consider the function

$$U_{\alpha,m}(x, y) = U_\alpha(x) - \sum_{|\lambda| \leqq m} \frac{x^\lambda}{\lambda!}[D^\lambda U_\alpha(-y)].$$

Here we give the estimates of $U_{\alpha,m}$ needed later.

Lemma 7.1. *If $y \in B(0, |x|/2)$, then*

$$|U_{\alpha,m}(x, y)| \leqq M|x|^m |y|^{\alpha-n-m}.$$

Lemma 7.2. *If $y \in B(0, 2|x|) - B(0, |x|/2)$, then*

$$|U_{\alpha,m}(x, y)| \leqq M|x - y|^{\alpha-n}.$$

Lemma 7.3. *If $y \in \mathbf{R}^n - B(0, 2|x|)$, then*

$$|U_{\alpha,m}(x, y)| \leqq M|x|^{m+1}|y|^{\alpha-n-m-1}.$$

Proof. Consider the function

$$\psi(t) = U_\alpha(tx - y).$$

By mean value theorem for analysis, we see that

$$\psi(1) = \psi(0) + (1/1!)\psi'(0) + \cdots + (1/m!)\psi^{(m)}(0) + K_{m+1},$$

where $K_{m+1} = [1/(m + 1)!]\psi^{(m+1)}(t_0)$ for some $0 < t_0 < 1$. Further, note that

$$\psi^{(\ell)}(t) = \sum_{|\lambda|=\ell} \frac{x^\lambda}{\lambda!}[D^\lambda U_\alpha(tx - y)].$$

If $y \in \mathbf{R}^n - B(0, 2|x|)$, then

$$|t_0 x - y| \geqq |y| - |x| \geqq |y|/2,$$

so that

$$|D^\lambda U_\alpha(t_0x - y)| \leqq M|y|^{\alpha-n-|\lambda|}.$$

Hence we obtain the required inequality.

THEOREM 7.1. *Let m be a positive integer smaller than α, and let f be a function in $L^p(\mathbf{R}^n)$ satisfying* (3.1). *If $J_{\alpha-m,p}(x_0) < \infty$ and $U_{\alpha-m}|f|(x_0) < \infty$, then there exist a set E and a polynomial P for which E is (α,p)-thin at x_0 and*

$$\lim_{x\to x_0, x\in\mathbf{R}^n-E} |x-x_0|^{-m}[U_\alpha f(x) - P(x)] = 0. \tag{7.1}$$

If (7.1) holds, then we say that $U_\alpha f$ is (α,p)-finely differentiable at x_0.

COROLLARY 7.1. *If f is as in Theorem 7.1, then $U_\alpha f$ is (α,p)-finely differentiable $(\alpha-m,p)$-q.e. on $\mathbf{R}^n$.*

PROOF OF THEOREM 7.1. For simplicity, assume that $x_0 = 0$ and $f \geqq 0$. For $x \neq 0$, write

$$\begin{aligned} U_{\alpha,m}f(x) &\equiv \int U_{\alpha,m}(x,y)f(y)dy \\ &= \int_{B(0,|x|/2)} U_{\alpha,m}(x,y)f(y)dy \\ &\quad + \int_{B(0,2|x|)-B(0,|x|/2)} U_{\alpha,m}(x,y)f(y)dy \\ &\quad + \int_{\mathbf{R}^n-B(0,2|x|)} U_{\alpha,m}(x,y)f(y)dy \\ &= u_1(x) + u_2(x) + u_3(x). \end{aligned}$$

By Lemma 7.1, we have

$$|u_1(x)| \leqq M|x|^m \int_{B(0,|x|/2)} |y|^{\alpha-n-m} f(y)dy,$$

so that

$$\lim_{x\to 0} |x|^{-m} u_1(x) = 0.$$

For $\varepsilon > 0$, set

$$\delta(\varepsilon) = \sup_{0<r<\varepsilon} \int_{B(0,r)} |y|^{\alpha-n-m} f(y)dy.$$

Then $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$. By Lemma 7.3, we have

$$|u_3(x)| \leqq M|x|^{m+1} \int_{\mathbf{R}^n-B(0,2|x|)} |y|^{\alpha-n-m-1} f(y)dy,$$

so that

$$\begin{aligned}|u_3(x)| &\leqq M|x|^{m+1}\int_{\mathbf{R}^n-B(0,\varepsilon)} |y|^{\alpha-n-m-1}f(y)dy \\ &\quad +M|x|^{m+1}\int_{2|x|}^{\varepsilon} r^{-1}d\left(\int_{B(0,r)} |y|^{\alpha-n-m}f(y)dy\right) \\ &\leqq M|x|^{m+1}\int_{\mathbf{R}^n-B(0,\varepsilon)} |y|^{\alpha-n-m-1}f(y)dy + M\delta(\varepsilon)|x|^m,\end{aligned}$$

so that

$$\limsup_{x\to 0} |x|^{-m}|u_3(x)| \leqq M\delta(\varepsilon).$$

Thus it follows that

$$\lim_{x\to 0} |x|^{-m}u_3(x) = 0.$$

To deal with u_2, in view of $J_{\alpha-m,p}(0) < \infty$, we take a sequence $\{a_j\}$ such that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j \left(2^{j[n-(\alpha-m)p]}\int_{B_j} f(y)^p dy\right)^{1/(p-1)} < \infty. \tag{7.2}$$

Now consider the sets

$$E_j = \{x : 2^{-j} \leqq |x| < 2^{-j+1}, |u_2(x)| > a_j^{-1/p'}|x|^m\}.$$

If $x \in E_j$ and $y \in B(0, 2|x|) - B(0, |x|/2)$, then Lemma 7.2 gives

$$|u_2(x)| \leqq M\int_{B_j} |x-y|^{\alpha-n}f(y)dy,$$

so that

$$C_{\alpha,p}(E_j; B_j) \leqq a_j^{p/p'}2^{jmp}\int_{B_j} f(y)^p dy.$$

Hence it follows from (7.2) that

$$\sum_{j=1}^{\infty}\left(2^{j(n-\alpha p)}C_{\alpha,p}(E_j; B_j)\right)^{1/(p-1)} < \infty.$$

This implies that $E = \bigcup_{j=1}^{\infty} E_j$ is (α, p)-thin at 0. On the other hand,

$$\limsup_{x\to 0, x\in\mathbf{R}^n-E} |x|^{-m}|u_2(x)| \leqq M\limsup_{j\to\infty} a_j^{-1/p'} = 0.$$

Thus $|x|^{-m}U_{\alpha,m}f(x)$ has (α, p)-fine limit zero at the origin, and the proof is completed.

Theorem 7.2. *Let $\alpha = m$ be a positive integer, and let f be a function in $L^p(\mathbf{R}^n)$ satisfying* (3.1). *Then for almost every x_0, there exist a set E and a polynomial P for which*

$$\lim_{x \to x_0, x \in \mathbf{R}^n - E} |x - x_0|^{-m}[U_\alpha f(x) - P(x)] = 0$$

and

$$\lim_{j \to \infty} 2^{j(n-\alpha p)} C_{\alpha,p}(E_j; B_j) = 0. \tag{7.3}$$

If (7.3) holds, then we say that E is (α, p)-semithin at x_0. We also say that $U_m f$ is (α, p)-semifinely differentiable at almost every point of $\mathbf{R}^n$.

To show Theorem 7.2, we need some lemmas.

Lemma 7.4. *The potential $\int_B |x - y|^{m-n} dy$ is infinitely differentiable inside a ball B.*

Proof. Let $\psi \in C_0^\infty(B)$ which is equal to 1 on $B(0, r_0)$, $0 < r_0 < 1$, and write

$$\begin{aligned} \int_B |x - y|^{m-n} dy &= \int |x - y|^{m-n} \psi(y) dy + \int_B |x - y|^{m-n}[1 - \psi(y)] dy \\ &= u_1(x) + u_2(x). \end{aligned}$$

Then it is easy to see that u_2 is infinitely differentiable inside $B(0, r_0)$ and u_1 is infinitely differentiable everywhere on $\mathbf{R}^n$. Since r_0 is arbitrary, the potential under consideration is infinitely differentiable on B.

Lemma 7.5. *Let f be a locally integrable function on $\mathbf{R}^n$ such that*

$$\int_{\mathbf{R}^n} (1 + |y|)^{-n} |f(y)| dy < \infty.$$

For $|\lambda| = m$,

$$A_\lambda(x) = \lim_{r \to 0} \int_{\mathbf{R}^n - B(x,r)} [D^\lambda |x - y|^{m-n}] f(y) dy$$

exists for almost every x.

This is a consequence of singular integral theory, whose proof will be given in Theorem 3.5 of Chapter 6.

Proof of Theorem 7.2. By Lemma 7.4, the function

$$U(x) = \int_{B(x_0,1)} |x - y|^{m-n} dy$$

is infinitely differentiable on $B(x_0, 1)$ and $B_\lambda = D^\lambda U(x_0)$ is independent of x_0. Set

$$C_\lambda = \begin{cases} A_\lambda(x_0) & \text{if } |\lambda| < m, \\ A_\lambda(x_0) + f(x_0) B_\lambda & \text{if } |\lambda| = m \end{cases}$$

and

$$P(x) = \sum_{|\lambda| \leqq m} \frac{C_\lambda}{\lambda!}(x - x_0)^\lambda.$$

For $x \in B(x_0, 1/2) - \{x_0\}$, write

$$\begin{aligned}
& |x - x_0|^{-m}\{U_m f(x) - P(x)\} \\
= \; & |x - x_0|^{-m} \int_{\mathbf{R}^n - B(x_0,1)} U_{m,m}(x - x_0, y - x_0) f(y) dy \\
& + |x - x_0|^{-m} \int_{B(x_0,1) - B(x_0, 2|x - x_0|)} U_{m,m}(x - x_0, y - x_0) f(y) dy \\
& - |x - x_0|^{-m} \sum_{|\lambda| \leqq m} \frac{(x - x_0)^\lambda}{\lambda!} \\
& \qquad \times \lim_{r \to 0} \int_{B(x_0, 2|x - x_0|) - B(0,r)} D^\lambda U_m(x_0 - y)\{f(y) - f(x_0)\} dy \\
& + f(x_0)|x - x_0|^{-m} \\
& \quad \times \left(\lim_{r \to 0} \int_{B(x_0,1) - B(0,r)} U_{m,m}(x - x_0, y - y_0) dy - \sum_{|\lambda| = m} \frac{B_\lambda}{\lambda!}(x - x_0)^\lambda \right) \\
& + |x - x_0|^{-m} \int_{B(x_0, 2|x - x_0|) - B(x, |x - x_0|/2)} U_m(x - y)\{f(y) - f(x_0)\} dy \\
& + |x - x_0|^{-m} \int_{B(x, |x - x_0|/2)} U_m(x - y)\{f(y) - f(x_0)\} dy \\
= \; & u_1(x) + u_2(x) - u_3(x) + u_4(x) + u_5(x) + u_6(x).
\end{aligned}$$

Lemma 7.3 implies that

$$\lim_{x \to x_0} u_1(x) = 0.$$

As in the proof of Theorem 7.1, set for $r > 0$,

$$\delta(\varepsilon) = \sup_{0 < t < \varepsilon} \frac{1}{|B(x_0, t)|} \int_{B(x_0,t)} |f(y) - f(x_0)| dy.$$

Note that

$$\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0 \tag{7.4}$$

for almost every x_0. Now assume that (7.4) holds. By Lemma 7.3, we have

$$\begin{aligned}
|u_2(x)| & \leqq M|x - x_0| \int_{B(x_0,1) - B(x_0, 2|x - x_0|)} |x_0 - y|^{-n-1} |f(y) - f(x_0)| dy \\
& \leqq M|x - x_0| \int_{B(x_0,1) - B(x_0,\varepsilon)} |x_0 - y|^{-n-1} |f(y) - f(x_0)| dy \\
& \quad + M|x - x_0| \int_{B(x_0,\varepsilon) - B(x_0, 2|x - x_0|)} |x_0 - y|^{-n-1} |f(y) - f(x_0)| dy \\
& \leqq M(\varepsilon)|x - x_0| + M\delta(\varepsilon).
\end{aligned}$$

Hence

$$\limsup_{x\to x_0} |u_2(x)| \leqq M\delta(\varepsilon),$$

which proves

$$\lim_{x\to x_0} u_2(x) = 0.$$

Since A_λ exists and is finite for $|\lambda| = m$,

$$\begin{aligned} &\lim_{r\to 0}\int_{B(x_0,2|x-x_0|)-B(x_0,r)} D^\lambda U_m(x_0-y)\{f(y)-f(x_0)\}dy \\ = &\lim_{r\to 0}\int_{B(x_0,2|x-x_0|)-B(x_0,r)} D^\lambda U_m(x_0-y)f(y)dy \end{aligned}$$

tends to zero as $x \to x_0$. If $|\lambda| < m$, then

$$\begin{aligned} &|x-x_0|^{-m}|(x-x_0)^\lambda|\left|\int_{B(x_0,2|x-x_0|)} D^\lambda U_m(x_0-y)\{f(y)-f(x_0)\}dy\right| \\ \leqq\ &M|x-x_0|^{-m+|\lambda|}\int_{B(x_0,2|x-x_0|)} |x_0-y|^{m-n-|\lambda|}|f(y)-f(x_0)|dy \leqq M\delta(2|x-x_0|). \end{aligned}$$

Thus it follows that

$$\lim_{x\to x_0} u_3(x) = 0.$$

Note that

$$u_4(x) = f(x_0)|x-x_0|^{-m}\left(U(x) - \sum_{|\lambda|\leqq m}\frac{(x-x_0)^\lambda}{\lambda!}D^\lambda U(x_0)\right).$$

Since U is infinitely differentiable near x_0,

$$\lim_{x\to x_0} u_4(x) = 0.$$

As to u_5, if we note that

$$|u_5(x)| \leqq M|x-x_0|^{-n}\int_{B(x_0,2|x-x_0|)} |f(y)-f(x_0)|dy,$$

then

$$\lim_{x\to x_0} u_5(x) = 0.$$

Finally we see as in the proof of Theorem 7.1 that

$$|u_6(x)| \leqq M|x-x_0|^{-m}\int_{B(x,|x-x_0|/2)} |x-y|^{m-n}|f(y)-f(x_0)|dy$$

has (m,p)-semifine limit zero at x_0.

5.8 Differentiability

Let φ be as in Section 5.3, and recall that

$$\Phi_p(r) = r^p \varphi(r).$$

For the positive integer m such that $m \leqq \alpha - n/p < m+1$, assuming that

$$\int_0^1 [r^{n-(\alpha-m)p}\varphi(r^{-1})]^{-p'/p} \frac{dr}{r} < \infty, \tag{8.1}$$

we define

$$\kappa_m(r) = \left(\int_0^r [t^{n-(\alpha-m)p}\varphi(t^{-1})]^{-p'/p} \frac{dt}{t} \right)^{1/p'}.$$

THEOREM 8.1. *Let m be as above, and let f be a function on $\mathbf{R}^n$ satisfying* (3.1) *and* (3.2). *Then $U_\alpha f$ is m times differentiable at any $x_0 \in \mathbf{R}^n$, and in fact, for some polynomial P of degree at most m,*

$$|x-x_0|^{-m}[U_\alpha f(x) - P(x)] = O(\kappa_m(|x-x_0|)) \qquad \text{as } x \to x_0.$$

PROOF. Without loss of generality, we may assume that $x_0 = 0$, $f \geqq 0$ on $\mathbf{R}^n$ and $f = 0$ outside a ball B. As in the proof of Theorem 7.1, write

$$U_{\alpha,m} f(x) = u_1(x) + u_2(x) + u_3(x).$$

Recalling the considerations in the proof of Theorem 3.1, we have for $0 < \gamma < \alpha - m < 1 + \gamma$,

$$\begin{aligned}
|x|^{-m}|u_1(x)| &\leqq M \int_{B(0,|x|/2)} |y|^{\alpha-n-m} f(y)dy \\
&= M \int_{\{y \in B(0,|x|/2): f(y) \leqq |y|^{-\gamma}\}} |y|^{\alpha-n-m} f(y)dy \\
&\quad + M \int_{\{y \in B(0,|x|/2): f(y) > |y|^{-\gamma}\}} |y|^{\alpha-n-m} f(y)dy \\
&\leqq M \int_{B(0,|x|/2)} |y|^{\alpha-n-m-\gamma} dy \\
&\quad + M \left(\int_{B(0,|x|/2)} [|y|^{\alpha-n-m}\varphi(|y|^{-\gamma})^{-1/p}]^{p'} dy \right)^{1/p'} \\
&\qquad \times \left(\int_{B(0,|x|/2)} \Phi_p(f(y))dy \right)^{1/p} \\
&\leqq M|x|^{\alpha-m-\gamma} + M\kappa_m(|x|) \left(\int_{B(0,|x|/2)} \Phi_p(f(y))dy \right)^{1/p},
\end{aligned}$$

$$
\begin{aligned}
|x|^{-m}|u_2(x)| &\leqq M|x|^{-m}\int_{B(0,2|x|)} |x-y|^{\alpha-n}f(y)dy \\
&\leqq M|x|^{-m+\alpha-\gamma} + M|x|^{-m}\left(\int_0^{2|x|} [r^{n-\alpha p}\varphi(r^{-\gamma})]^{-p'/p}r^{-1}dr\right)^{1/p'} \\
&\qquad\qquad \times\left(\int_{B(0,2|x|)} \Phi_p(f(y))dy\right)^{1/p} \\
&\leqq M|x|^{\alpha-m-\gamma} + M\kappa_m(|x|)\left(\int_{B(0,2|x|)} \Phi_p(f(y))dy\right)^{1/p}
\end{aligned}
$$

and

$$
\begin{aligned}
|x|^{-m}|u_3(x)| &\leqq M|x|\int_{B-B(0,2|x|)} |y|^{\alpha-n-m-1}f(y)dy \\
&\leqq M|x|^{\alpha-m-\gamma} + M|x|\left(\int_B \Phi_p(f(y))dy\right)^{1/p} \\
&\qquad\qquad \times\left(\int_{2|x|}^{\infty}[r^{n-(\alpha-m-1)p}\varphi(r^{-1})]^{-p'/p}dr\right)^{1/p'} \\
&\leqq M|x|^{\alpha-m-\gamma} + M[|x|^{n-(\alpha-m)p}\varphi(|x|^{-1})]^{-1/p}\left(\int_B \Phi_p(f(y))dy\right)^{1/p}.
\end{aligned}
$$

Here note that

$$\kappa_m(r) \geqq M[r^{n-(\alpha-m)p}\varphi(r^{-1})]^{-1/p}. \tag{8.2}$$

Hence the required result follows.

Remark 8.1. If $n-(\alpha-m)p=0$, then

$$|x-x_0|^{-m}[U_\alpha f(x)-P(x)] = o(\kappa_m(|x-x_0|)) \qquad \text{as } x\to x_0,$$

because, in this case, (8.2) is replaced by

$$\kappa_m(r) \geqq \left(\int_{r^2}^{r} [\varphi(t^{-1})]^{-p'/p}\frac{dt}{t}\right)^{1/p'} \geqq M[\varphi(r^{-1})]^{-1/p}[\log(1/r)]^{1/p'}.$$

Next we relax (8.1), and in fact assume that

$$\int_0^1 [r^{n-\alpha p}\varphi(r^{-1})]^{-p'/p}\frac{dr}{r} < \infty. \tag{8.3}$$

For a positive integer m, define

$$h_m(r) = \inf_{t\geqq r} t^{mp}\left(\int_0^t [s^{n-\alpha p}\varphi(s^{-1})]^{-p'/p}\frac{ds}{s}\right)^{-p/p'}.$$

Here note that

$$h_m(r) \leqq M r^{n-(\alpha-m)p} \varphi(r^{-1}) \tag{8.4}$$

and, in case $\alpha p > n$,

$$h_m(r) \geqq M r^{n-(\alpha-m)p} \varphi(r^{-1}). \tag{8.5}$$

For an open set G and $\beta > 0$, we define

$$C_{\beta,\Phi_p}(E;G) = \inf \int_G \Phi_p(f(y))dy,$$

where the infimum is taken over all nonnegative measurable functions f such that

$$\int_G |x-y|^{\beta-n} f(y)dy \geqq 1 \qquad \text{for all } x \in E.$$

As before, write $C_{\beta,\Phi_p}(E) = 0$ if

$$C_{\beta,\Phi_p}(E \cap G; G) = 0 \qquad \text{for any bounded open set } G.$$

LEMMA 8.1. *If f satisfies* (3.2), *then*

$$C_{\alpha-m,\Phi_p}(E_f) = 0,$$

where

$$E_f = \left\{ x : \int_{B(x,1)} |x-y|^{\alpha-n-m} |f(y)| dy = \infty \right\}.$$

LEMMA 8.2. *Let h be a measure function on $[0,\infty)$ for which*

$$\lim_{r \to 0} r^{-n} h(r) = \infty,$$

and set

$$E_{f,h} = \left\{ x : \limsup_{r \to 0} \, [h(r)]^{-1} \int_{B(x,r)} |f(y)| dy > 0 \right\}.$$

If f is locally integrable, then $H_h(E_{f,h}) = 0$.

PROOF. For $\delta > 0$, we have only to show that

$$H_h(E_{f,h,\delta} \cap B(0,R)) = 0 \qquad \text{for any } R > 0,$$

where

$$E_{f,h,\delta} = \left\{ x : \limsup_{r \to 0} \, [h(r)]^{-1} \int_{B(x,r)} |f(y)| dy > \delta \right\}.$$

Let $\varepsilon > 0$. For each $x \in E_{f,h,\delta} \cap B(0,R)$, we can find $r(x)$ such that $0 < r(x) < \varepsilon$ and

$$\int_{B(x,r(x))} |f(y)|dy > \delta h(r(x)).$$

By a covering lemma (see Theorem 10.1 in Chapter 1), we can select a mutually disjoint family $\{B(x_j, r_j)\}$ for which $r_j = r(x_j)$ and

$$\bigcup_j B(x_j, 5r_j) \supseteqq E_{f,h,\delta} \cap B(0,R).$$

Hence we see that

$$\begin{aligned} H_h^{(5\varepsilon)}(E_{f,h,\delta} \cap B(0,R)) &\leqq \sum_j h(5r_j) \leqq M \sum_j h(r_j) \\ &\leqq M\delta^{-1} \sum_j \int_{B(x_j,r_j)} |f(y)|dy \\ &\leqq M\delta^{-1} \int_{\cup_j B(x_j,r_j)} |f(y)|dy. \end{aligned}$$

On the other hand,

$$\int_{B(x_j,r_j)} |f(y)|dy > \delta h(r_j) = \delta \frac{h(r_j)}{{r_j}^n} {r_j}^n,$$

so that

$$\begin{aligned} |\bigcup_j B(x_j, r_j)| &= \sum_j |B(x_j, r_j)| \\ &\leqq M\delta^{-1} [\inf_{0<r<\varepsilon} h(r)/r^n]^{-1} \int_{B(0,R+5\varepsilon)} f(y)dy. \end{aligned}$$

Here the last term tends to 0 as $\varepsilon \longrightarrow 0$, and thus by considering the absolute continuity with respect to the n-dimensional measure,

$$\int_{\cup_j B(x_j,r_j)} |f(y)|dy$$

becomes small with ε. Hence it follows that

$$H_h(E_{f,h,\delta} \cap B(0,R)) = 0,$$

as required.

THEOREM 8.2. *Let m be a positive integer such that $\alpha - n/p \leqq m < \alpha$, and let f be a function on $\mathbf{R}^n$ satisfying* (3.1) *and* (3.2). *Then there exist E_1 and E_2 such that $C_{\alpha-m,\Phi_p}(E_1) = 0$, $H_{h_m}(E_2) = 0$ and $U_\alpha f$ is m times differentiable at any $x \in \mathbf{R}^n - (E_1 \cup E_2)$.*

PROOF. For simplicity, let $E_1 = E_f$ and $E_2 = E_{f,h}$ with f and h replaced by $\Phi_p(|f(y)|)$ and h_m, respectively. We show below that E_1 and E_2 satisfy the required assertions. In view of Lemmas 8.1 and 8.2,

$$C_{\alpha-m,\Phi_p}(E_1) = 0 \qquad \text{and} \qquad H_{h_m}(E_2) = 0.$$

Now assume that $x_0 \notin E_1 \cup E_2$. As in the above proof, write

$$U_{\alpha,m}f(x) = u_1(x) + u_2(x) + u_3(x).$$

First we see that

$$|x - x_0|^{-m}|u_1(x)| \leqq M \int_{B(x_0,|x-x_0|/2)} |x_0 - y|^{\alpha-n-m}|f(y)|dy$$

tends to zero as $x \to x_0$, since $x_0 \notin E_1$. Similarly, since

$$\lim_{r\to 0} \int_{B(x_0,r)} |x_0 - y|^{\alpha-n-m}|f(y)|dy = 0,$$

it follows that

$$\begin{aligned} &\limsup_{x\to x_0} |x - x_0|^{-m}|u_3(x)| \\ \leqq\ & M \limsup_{x\to x_0} |x - x_0| \int_{\mathbf{R}^n - B(x_0,2|x-x_0|)} |x_0 - y|^{\alpha-n-m-1}|f(y)|dy = 0. \end{aligned}$$

Finally, we have for $0 < \gamma < \alpha$,

$$\begin{aligned} |x - x_0|^{-m}|u_2(x)| &\leqq M|x - x_0|^{-m} \int_{B(x_0,2|x-x_0|)} |x - y|^{\alpha-n} f(y)dy \\ &\leqq M|x - x_0|^{-m+\alpha-\gamma} \\ &\quad +M\left([h_m(|x - x_0|)]^{-1} \int_{B(x_0,2|x-x_0|)} \Phi_p(|f(y)|)dy\right)^{1/p}. \end{aligned}$$

Since $x_0 \notin E_2$,

$$\lim_{x\to x_0} |x - x_0|^{-m}u_2(x) = 0.$$

Thus the required assertion follows.

By applying the proofs of Theorems 7.2 and 8.2, we can prove the following result.

THEOREM 8.3. *Let $m = \alpha$ be a positive integer, and let f be a function on $\mathbf{R}^n$ satisfying* (3.1) *and* (3.2). *Then $U_\alpha f$ is m times differentiable almost everywhere on $\mathbf{R}^n$.*

THEOREM 8.4. *Let m be a positive integer smaller than α, and let f be a function in $L^p(\mathbf{R}^n)$ satisfying* (3.1). *If $U_{\alpha-m}|f|(x_0) < \infty$ and*

(8.6) $$\lim_{r\to 0} r^{-[n-(\alpha-m)p]} \int_{B(x_0,r)} |f(y)|^p dy = 0,$$

then there exists a polynomial P for which

$$\lim_{r\to 0} r^{-m}\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |U_\alpha f(x)-P(x)|^q dx\right)^{1/q} = 0 \tag{8.7}$$

whenever $1/q \geqq 1/p^* = 1/p - \alpha/n > 0$ and $q > 1$.

PROOF. Write $U_{\alpha,m}f$ as in the proofs of Theorems 7.1 and 8.2. Then it suffices to treat only u_2. By Lemma 7.2, we have

$$\begin{aligned} |u_2(x)| &\leqq M\int_{B(0,2|x|)} |x-y|^{\alpha-n}|f(y)|dy \\ &= Mr^\alpha \int_{B(0,2)} |z-w|^{\alpha-n}|f(rw)|dw \end{aligned}$$

for $r = |x|$. Hence it follows from Sobolev's theorem that

$$\begin{aligned} \left(\frac{1}{|B(0,r)|}\int_{B(0,r)} |u_2(x)|^q dx\right)^{1/q} &= \left(\frac{1}{|\mathbf{B}|}\int_{\mathbf{B}} |u_2(rz)|^q dz\right)^{1/q} \\ &\leqq Mr^\alpha\left(\int_{B(0,2)} |f(rw)|^p dw\right)^{1/p} \\ &\leqq Mr^{\alpha-n/p}\left(\int_{B(0,2r)} |f(y)|^p dy\right)^{1/p}. \end{aligned}$$

Consequently,

$$r^{-m}\left(\frac{1}{|B(0,r)|}\int_{B(0,r)}|u_2(x)|^q dx\right)^{1/q} \leqq M\left(r^{-[n-(\alpha-m)p]}\int_{B(0,2r)}|f(y)|^p dy\right)^{1/p},$$

which tends to zero as $r \to 0$ with the aid of (8.6).

COROLLARY 8.1. *If f is as above, then*

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} |U_\alpha f(y)-U_\alpha f(x)|^q dy = 0 \quad \textit{for } (\alpha,p)\textit{-q.e. } x\in \mathbf{R}^n,$$

whenever $1/q \geqq 1/p^* = 1/p - \alpha/n > 0$ and $q > 1$.

We say that a function u is m times L^q-mean differentiable at x_0 if there exists a polynomial P of degree at most m such that

$$\lim_{r\to 0} r^{-m}\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |u(y)-P(y)|^q dy\right)^{1/q} = 0. \tag{8.8}$$

COROLLARY 8.2. *If f, m and q are as in Theorem 8.4, then $U_\alpha f$ is m times L^q-mean differentiable $(\alpha-m,p)$-q.e. on $\mathbf{R}^n$.*

5.9 Logarithmic potentials

For a locally integrable function f on $\mathbf{R}^n$, we define the logarithmic potential by

$$Lf(x) = \int \left(\log \frac{1}{|x-y|}\right) f(y)dy.$$

Here assume that

$$\int [\log(2+|y|)]|f(y)|dy < \infty. \tag{9.1}$$

If (9.1) holds, then

$$-\infty < Lf \not\equiv \infty.$$

THEOREM 9.1. *Let f be a function on $\mathbf{R}^n$ satisfying* (9.1). *If*

$$\int [\log(2+|f(y)|)]|f(y)|dy < \infty, \tag{9.2}$$

then Lf is continuous on $\mathbf{R}^n$.

PROOF. For $r > 0$, we have

$$\begin{aligned} & \left| \int_{B(x_0,r)} [\log|x-y|] f(y)dy \right| \\ \leqq & \left| \int_{B(x_0,r)} [\log|x-y|] |x-y|^{-1} dy \right| + \int_{B(x_0,r)} |[\log|f(y)|] f(y)|dy \\ \leqq & \ M r^{n-1} \log(1/r) + \int_{B(x_0,r)} |[\log|f(y)|] f(y)|dy. \end{aligned}$$

This implies that $Lf(0)$ is finite and

$$\lim_{x\to x_0} Lf(x) = Lf(x_0).$$

THEOREM 9.2. *Let f be a function on $\mathbf{R}^n$ satisfying* (9.1) *and* (9.2). *Then Lf is n times differentiable almost everywhere on $\mathbf{R}^n$.*

THEOREM 9.3. *Let $1 < p < \infty$ and f be a function in $L^p(\mathbf{R}^n)$ satisfying* (9.1). *If m is a positive integer smaller than n, then Lf is m times differentiable $(n-m,p)$-q.e. on $\mathbf{R}^n$. Moreover, Lf is n times differentiable a.e. on $\mathbf{R}^n$ and the n-th derivatives belong to $L^p(\mathbf{R}^n)$.*

The last assertion follows from the singular integral theory given in the next chapter.

Chapter 6

Beppo Levi functions

Beppo Levi functions are functions whose distributional derivatives are (locally) in L^p. Beppo Levi functions can be represented as integral forms, in various ways. If this is done, then Beppo Levi functions are seen to behave like potentials of functions in L^p. To show the converse, we apply the singular integral theory, whose proof is also given here.

6.1 Sobolev's integral representation

We begin with the following lemma, which is needed later to establish an integral representation for functions whose partial derivatives are all in the Lebesgue class $L^p(\mathbf{R}^n)$.

LEMMA 1.1. *Let u be a locally integrable function on a domain G. If $D^\lambda u = 0$ on G for any multi-index λ with length m in the sense of distributions, then u is equal a.e. on G to a polynomial of degree at most $m-1$.*

PROOF. Let ψ be a nonnegative function in $C_0^\infty(\mathbf{B})$ for which

$$\int \psi(x)dx = 1,$$

and set

$$\psi_\delta(x) = \delta^{-n}\psi(x/\delta)$$

for $\delta > 0$. Then $u * \psi_\delta$ is defined on $G_\delta = \{x \in G : \mathrm{dist}(x, \partial G) > \delta\}$ and

$$D^\lambda(u * \psi_\delta) = (D^\lambda u) * \psi_\delta = 0$$

there for $|\lambda| = m$. This implies that $u * \psi_\delta$ is equal to a polynomial of degree at most $m-1$ on each component of G_δ. If $m = 1$, then, as seen above, $u * \psi_\delta$ is constant on each component of G_δ. Since $u * \psi_\delta \to u$ in $L^1_{loc}(G)$, we see that u is constant a.e. on G. If $m = 2$, then each partial derivative $(\partial/\partial x_j)u$ is constant a_j a.e. on G, so that

$u(x_1, ..., x_n) - \sum_j a_j x_j$ is shown to be constant a.e. on G. In general we see that u is equal a.e. on G to a polynomial of degree at most $m-1$.

THEOREM 1.1 (Sobolev's integral representation). *If $\psi \in C_0^\infty(\mathbf{R}^n)$, then*

$$\psi(x) = \sum_{|\lambda|=m} a_\lambda \int \frac{(x-y)^\lambda}{|x-y|^n} D^\lambda \psi(y) dy, \qquad a_\lambda = \frac{m}{\omega_n \lambda!}.$$

PROOF. For $\Theta \in \mathbf{S} = S(0,1)$,

$$\psi(0) = \frac{(-1)^m}{(m-1)!} \int_0^\infty t^{m-1} (\partial/\partial t)^m \psi(t\Theta) dt.$$

Note that

$$\frac{1}{m!} (\partial/\partial t)^m \psi(t\Theta) = \sum_{|\lambda|=m} \frac{\Theta^\lambda}{\lambda!} D^\lambda \psi(t\Theta).$$

Hence, integrating with respect to $\Theta \in \mathbf{S}$, we have

$$\begin{aligned} \omega_n \psi(0) &= \frac{(-1)^m}{(m-1)!} \int_{\mathbf{S}} \left(\int_0^\infty t^{m-1} (\partial/\partial t)^m \psi(t\Theta) dt \right) d\Theta \\ &= (-1)^m m \sum_{|\lambda|=m} \frac{1}{\lambda!} \int \frac{y^\lambda}{|y|^n} D^\lambda \psi(y) dy. \end{aligned}$$

Applying this with $\psi(x - \cdot)$, we have the required equality.

Let $u \in L^p_{loc}(G)$. If $D^\lambda u \in L^p(G)$ for any multi-index λ with length m, then u is called a Beppo Levi function on G and written as

$$u \in BL_m(L^p(G)).$$

We show that Sobolev's integral representation is valid for Beppo Levi functions. First we give a simple case which is derived from a general case mentioned later.

THEOREM 1.2. *Let $mp < n$. If $u \in BL_m(L^p(\mathbf{R}^n))$, then*

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int \frac{(x-y)^\lambda}{|x-y|^n} D^\lambda u(y) dy + P(x) \qquad \text{a.e. on } \mathbf{R}^n$$

for some polynomial P of degree at most $m-1$.

For a multi-index λ and a nonnegative integer ℓ, set

$$k_\lambda(x) = x^\lambda / |x|^n$$

and

$$k_{\lambda,\ell}(x,y) = \begin{cases} k_\lambda(x-y) & \text{for } |y| < 1, \\ k_\lambda(x-y) - \sum_{|\nu| \leqq \ell} \frac{x^\nu}{\nu!}[(D^\nu k_\lambda)(-y)] & \text{for } |y| \geqq 1. \end{cases}$$

By Lemma 7.3 in Chapter 5, we have the following estimate.

LEMMA 1.2. If $|y| \geqq 1$ and $|y| > 2|x|$, then

$$|k_{\lambda,\ell}(x,y)| \leqq M|x|^{\ell+1}|y|^{|\lambda|-n-\ell-1}.$$

THEOREM 1.3. *Let ℓ be the nonnegative integer such that $\ell \leqq m - n/p < \ell+1$. If $u \in BL_m(L^p(\mathbf{R}^n))$, then*

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int k_{\lambda,\ell}(x,y) D^\lambda u(y)dy + P(x) \qquad \text{a.e. on } \mathbf{R}^n$$

for some polynomial P of degree at most $m-1$.

PROOF. Denote by U the sum on the right-hand side. For $f \in L^p(\mathbf{R}^n)$ and a ball B, note by Hölder's inequality that

$$\int_{\mathbf{R}^n - B} |y|^{m-n-\ell-1}|f(y)|dy < \infty$$

since $(m-n-\ell-1)p' + n < 0$. Hence the function

$$\int_{\mathbf{R}^n - B} k_{\lambda,\ell}(x,y) f(y)dy$$

is finite-valued and continuous on B. Moreover,

$$\int_B k_{\lambda,\ell}(x,y)f(y)dy = \int_B k_\lambda(x-y)f(y)dy + \text{a polynomial},$$

so that it is locally integrable on $\mathbf{R}^n$. Thus U is locally integrable on $\mathbf{R}^n$. Let μ be a multi-index with length $m+\ell+1$ and write $\mu = \mu_1 + \mu_2$ with $|\mu_1| = \ell+1$ and $|\mu_2| = m$. Then we have for $\psi \in C_0^\infty(\mathbf{R}^n)$

$$\int U(x) D^\mu \psi(x)dx = \sum_{|\lambda|=m} a_\lambda \int \left(\int k_{\lambda,\ell}(x,y) D^\mu \psi(x)dx \right) D^\lambda u(y)dy.$$

For each positive integer j, set

$$k_\lambda^{(j)}(x) = \frac{x^\lambda}{|x|^n + (1/j)}.$$

Since $\ell < |\mu_1|$, we have

$$\begin{aligned}\int k_{\lambda,\ell}(x,y)D^\mu\psi(x)dx &= \int k_\lambda(x-y)D^\mu\psi(x)dx \\ &= (-1)^{|\mu_1|}\lim_{j\to\infty}\int D^{\mu_1}k_\lambda^{(j)}(x-y)D^{\mu_2}\psi(x)dx \\ &= (-1)^{\ell+1}\lim_{j\to\infty}\int D^{\mu_1}k_\lambda^{(j)}(z)D^{\mu_2}\psi(z+y)dz.\end{aligned}$$

Noting that $|D^{\mu_1}k_\lambda^{(j)}(z)| \leqq M|z|^{m-n-|\mu_1|}$, we apply Fubini's theorem to establish

$$\begin{aligned}\int U(x)D^\mu\psi(x)dx &= (-1)^{\ell+1}\sum_{|\lambda|=m}a_\lambda\lim_{j\to\infty}\int D^{\mu_1}k_\lambda^{(j)}(z)\left(\int D^{\mu_2}\psi(z+y)D^\lambda u(y)dy\right)dz \\ &= (-1)^{\ell+1}\sum_{|\lambda|=m}a_\lambda\lim_{j\to\infty}\int D^{\mu_1}k_\lambda^{(j)}(z)\left(\int D^{\lambda}\psi(z+y)D^{\mu_2} u(y)dy\right)dz \\ &= (-1)^{\ell+1}\sum_{|\lambda|=m}a_\lambda\lim_{j\to\infty}\int\left(\int D^{\mu_1}k_\lambda^{(j)}(z)D^{\lambda}\psi(z+y)dz\right)D^{\mu_2} u(y)dy \\ &= \sum_{|\lambda|=m}a_\lambda\int\left(\int k_\lambda(z)D^{\lambda+\mu_1}\psi(z+y)dz\right)D^{\mu_2} u(y)dy \\ &= (-1)^m\int D^{\mu_1}\psi(y)D^{\mu_2}u(y)dy \\ &= \int u(y)D^\mu\psi(y)dy.\end{aligned}$$

Thus Lemma 1.1 implies that $u-U$ is equal almost everywhere to a polynomial P of degree at most $m+\ell$. If we apply a result from singular integral theory (as will be shown later), then we see that

$$D^\mu P \in L^p(\mathbf{R}^n) \qquad \text{for any } \mu \text{ with length } m.$$

This implies that $D^\mu P = 0$ for any μ with length m, so that the degree of P is at most $m-1$.

6.2 Canonical representation

For a number ℓ, note that

$$\Delta|x|^\ell = \ell(n+\ell-2)|x|^{\ell-2} \qquad \text{on } \mathbf{R}^n - \{0\}$$

and

$$\Delta(\log|x|) = (n-2)|x|^{-2} \qquad \text{on } \mathbf{R}^n - \{0\}.$$

In view of Theorem 2.9 in Chapter 2, if $2m < n$, then

$$\psi(x) = b_m\int |x-y|^{2m-n}\Delta^m\psi(y)dy$$

for every $\psi \in C_0^\infty(\mathbf{R}^n)$, where

$$b_m \equiv [\gamma_{2m}]^{-1}(-4\pi^2)^{-m} = (-1)^m \frac{\Gamma((n-2m)/2)}{(m-1)!4^m \pi^{n/2}}.$$

In general we show the following.

THEOREM 2.1. *If $\psi \in C_0^\infty(\mathbf{R}^n)$, then*

$$\psi(x) = b_m \int U_{2m}(x-y)\Delta^m \psi(y)dy,$$

where b_m is a constant and

$$U_{2m}(x) = \begin{cases} |x|^{2m-n} & \text{when } 2m < n \text{ or } 2m \geqq n \text{ and } n \text{ is odd,} \\ |x|^{2m-n}\log(1/|x|) & \text{when } 2m \geqq n \text{ and } n \text{ is even.} \end{cases}$$

By writing

$$\Delta^m = \sum_{|\lambda|=m} (m!/\lambda!)D^{2\lambda},$$

Theorem 2.1 establishes the canonical representation for functions in $C_0^\infty(\mathbf{R}^n)$.

LEMMA 2.1. *If $\psi \in C_0^\infty(\mathbf{R}^n)$, then*

$$\psi(x) = \sum_{|\lambda|=m} b_\lambda \int (D^\lambda U_{2m})(x-y)D^\lambda \psi(y)dy$$

with $b_\lambda = m!b_m/\lambda!$.

Setting $\tilde{k}_\lambda(x) = D^\lambda U_{2m}(x)$, we define $\tilde{k}_{\lambda,\ell}(x,y)$ as before Lemma 1.2.

THEOREM 2.2. *Let ℓ be the nonnegative integer such that $\ell \leqq m - n/p < \ell + 1$. If $u \in BL_m(L^p(\mathbf{R}^n))$, then*

$$u(x) = \sum_{|\lambda|=m} b_\lambda \int_{\mathbf{R}^n} \tilde{k}_{\lambda,\ell}(x,y)D^\lambda u(y)dy + P(x) \qquad \text{a.e. on } \mathbf{R}^n$$

for some polynomial P of degree at most $m-1$.

PROOF. For $\psi \in C_0^\infty(\mathbf{R}^n)$, we have

$$\int \left(\sum_{|\lambda|=m} b_\lambda \int_{\mathbf{R}^n} \tilde{k}_{\lambda,\ell}(x,y)D^\lambda u(y)dy \right) \Delta^m \psi(x)dx$$
$$= \sum_{|\lambda|=m} b_\lambda \int_{\mathbf{R}^n} \left(\int \tilde{k}_{\lambda,\ell}(x,y)\Delta^m \psi(x)dx \right) D^\lambda u(y)dy$$

$$
\begin{aligned}
&= \sum_{|\lambda|=m} b_\lambda \int_{\mathbf{R}^n} \left((-1)^m \int U_{2m}(x-y) \Delta^m (D^\lambda \psi(x)) dx \right) D^\lambda u(y) dy \\
&= (-1)^m b_m^{-1} \sum_{|\lambda|=m} b_\lambda \int_{\mathbf{R}^n} D^\lambda \psi(y) D^\lambda u(y) dy \\
&= b_m^{-1} \sum_{|\lambda|=m} b_\lambda \int_{\mathbf{R}^n} [D^{2\lambda} \psi(y)] u(y) dy \\
&= \int_{\mathbf{R}^n} u(y) \Delta^m \psi(y) dy,
\end{aligned}
$$

which proves the required assertion in view of the proof of Theorem 1.3.

6.3 Theory of singular integrals

First we show the so-called Calderón-Zygmund decomposition.

THEOREM 3.1. *Let f be a nonnegative integrable function on $\mathbf{R}^n$ and $t > 0$. Then there exist sets F and Ω such that*

(i) $\mathbf{R}^n = F \cup \Omega$, $F \cap \Omega = \emptyset$.

(ii) *$f(x) \leqq t$ for almost every $x \in F$.*

(iii) *Ω is the union of cubes $\{Q_j\}$ whose interiors are mutually disjoint.*

(iv) *$t < \frac{1}{|Q_j|} \int_{Q_j} f(x)dx \leqq 2^n t$ for all Q_j.*

PROOF. Since $f \in L^1(\mathbf{R}^n)$, we can find $r_0 > 0$ such that

$$\frac{1}{|Q|} \int_Q f(x)dx \leqq t \qquad \text{for any cube } Q \text{ with diameter } r_0.$$

Decompose $\mathbf{R}^n$ into a mesh of equal cubes with diameter r_0 whose interiors are mutually disjoint. Taking any cube Q' in this mesh, we divide it into 2^n congruent cubes $\{Q''\}$, and collect cubes Q'' satisfying

$$\frac{1}{|Q''|} \int_{Q''} f(x)dx > t; \tag{3.1}$$

here note that

$$t < \frac{1}{|Q''|} \int_{Q''} f(x)dx \leqq \frac{1}{2^{-n}|Q'|} \int_{Q'} f(x)dx \leqq 2^n t. \tag{3.2}$$

If Q'' does not satisfy (3.1), then we again divide it into 2^n congruent cubes $\{Q'''\}$, and collect cubes Q''' satisfying (3.1). Repeating the sub-division process, we finally obtain cubes $\{Q_j\}$ satisfying (3.2), and set

$$\Omega = \bigcup_j Q_j \qquad \text{and} \qquad F = \mathbf{R}^n - \Omega.$$

If $x \in F$, then there exists a sequence $\{Q_\ell(x)\}$ of cubes such that $x \in Q_\ell(x)$, $Q_{\ell+1}(x) \subseteqq Q_\ell(x)$, $\mathrm{diam}(Q_\ell(x)) = 2^{-\ell} r_0$ and

$$\frac{1}{|Q_\ell(x)|} \int_{Q_\ell(x)} f(y)dy \leqq t,$$

so that it follows that

$$f(x) \leqq t \qquad \text{for a.e. } x \in F.$$

REMARK 3.1. In Theorem 3.1, we have

$$|\Omega| = \sum_j |Q_j| < \frac{1}{t} \int_\Omega f(x)dx \leqq \frac{1}{t}\|f\|_1.$$

Applying Plancherel's theorem, we have the following result.

LEMMA 3.1. *Let $K \in \mathcal{S}'$ be a tempered distribution whose Fourier transform is bounded, that is,*

$$\hat{K} \in L^\infty(\mathbf{R}^n). \tag{3.3}$$

Then the convolution operator

$$Tf(x) = K * f(x), \qquad f \in \mathcal{S},$$

is of type $(2,2)$, that is,

$$\|Tf\|_2 \leqq A\|f\|_2 \qquad \text{for } f \in \mathcal{S};$$

thus Tf is well-defined for $f \in L^2(\mathbf{R}^n)$ so that

$$\mathcal{F}(Tf) = \hat{K}\hat{f}.$$

LEMMA 3.2. *Let K be a function in $L^1_{loc}(\mathbf{R}^n - \{0\})$ such that*

$$A \equiv \sup_y \int_{\{x:|x|>2|y|\}} |K(x-y) - K(x)| \, dx < \infty. \tag{3.4}$$

If a is a integrable function on a cube Q centered at the origin for which

$$\int_Q a(y) \, dy = 0.$$

Then the convolution

$$\begin{aligned} K * a(x) &= \int_Q K(x-y)a(y) \, dy \\ &= \int_Q [K(x-y) - K(x)]a(y) \, dy, \end{aligned}$$

makes sense for almost every $x \in \mathbf{R}^n - \tilde{Q}$*, where* $\tilde{Q} = 2\sqrt{n}Q$*, and*

$$\int_{\mathbf{R}^n - \tilde{Q}} |K * a(x)| \, dx \leqq A\|a\|_1.$$

In fact, applying Fubini's theorem, we have

$$\begin{aligned} \int_{\mathbf{R}^n - \tilde{Q}} |K * a(x)| \, dx &\leqq \int_{\mathbf{R}^n - \tilde{Q}} \left(\int_{\{x:|x|>2|y|\}} |K(x-y) - K(x)| dx \right) |a(y)| \, dy \\ &\leqq A\|a\|_1. \end{aligned}$$

Lemma 3.3. *Let* k *be a function in* $L^1_{loc}(\mathbf{R}^n - \{0\})$ *such that*

(3.5) $$\left| \int_{\{x:r<|x|<R\}} k(x) \, dx \right| < A_1 \qquad \text{whenever } 0 < r < R,$$

(3.6) $$\int_{\{x:r<|x|<2r\}} |k(x)| \, dx < A_2 \qquad \text{whenever } r > 0$$

and

(3.7) $$\lim_{r \to 0} \int_{\{x:r<|x|<1\}} k(x) \, dx \qquad \text{exists.}$$

Then

$$K(\varphi) = p.v. \int k(x)\varphi(x) \, dx = \lim_{r \to 0} \int_{\{x:|x|>r\}} k(x)\varphi(x) \, dx$$

defines a tempered distribution on $\mathbf{R}^n$*.*

Proof. Write

$$\begin{aligned} K(\varphi) &= \varphi(0) \lim_{r \to 0} \int_{\{x:r<|x|<1\}} k(x) \, dx \\ &\quad + \lim_{r \to 0} \int_{\{x:r<|x|<1\}} k(x)[\varphi(x) - \varphi(0)] \, dx \\ &\quad + \int_{\{x:|x|>1\}} k(x)\varphi(x) \, dx. \end{aligned}$$

Set

$$C_1 = \sup_{\{x:|x|<1\}} \frac{|\varphi(x) - \varphi(0)|}{|x|} \leqq \sup_{\{|x|<1\}} |\nabla \varphi(x)|$$

and

$$C_2 = \sup_{\{x:|x|>1\}} |x||\varphi(x)|.$$

Then it follows that

$$\begin{aligned}\int_{\{x:r<|x|<1\}} |k(x)||\varphi(x)-\varphi(0)|\ dx &\leqq C_1 \sum_{j=1}^{\infty} \int_{\{x:2^{-j}<|x|<2^{-j+1}\}} |x||k(x)|\ dx \\ &\leqq C_1 \sum_{j=1}^{\infty} (A_2 2^{-j+1}) = 2C_1A_2\end{aligned}$$

and

$$\begin{aligned}\int_{\{x:|x|>1\}} |k(x)||\varphi(x)|\ dx &\leqq C_2 \sum_{j=1}^{\infty} \int_{\{x:2^{j-1}<|x|<2^{j}\}} |x|^{-1}|k(x)|\ dx \\ &\leqq C_2 \sum_{j=1}^{\infty} (A_2 2^{-j+1}) = 2C_2A_2.\end{aligned}$$

THEOREM 3.2. *Let k be a function in $L^1_{loc}(\mathbf{R}^n - \{0\})$ satisfying (3.3) – (3.7). Then there exists a constant $M > 0$ such that*

$$|\{x : Tf(x) > t\}| \leqq M\frac{\|f\|_1}{t} \qquad \text{for all } t > 0 \text{ and all } f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n).$$

PROOF. For $t > 0$, letting $\Omega = \bigcup_j Q_j$ and F be as in Theorem 3.1, we define

$$g(x) = \begin{cases} f(x) & \text{for } x \in F, \\ \dfrac{1}{|Q_j|}\displaystyle\int_{Q_j} f(y)dy & \text{for } x \in \mathrm{Int}(Q_j); \end{cases}$$

note that g is defined almost everywhere. If we set $b = f - g$, then

$$b = 0 \qquad \text{a.e. on } F$$

and

$$\int_{Q_j} b(y)dy = 0 \qquad \text{for all cubes } Q_j. \tag{3.8}$$

Note that $g \in L^2(\mathbf{R}^n)$, because

$$\begin{aligned}\int_{\mathbf{R}^n} |g(y)|^2\ dy &= \int_F |f(y)|^2\ dy + \sum_j \int_{Q_j} |g(y)|^2\ dy \\ &\leqq \int_F t|f(y)|\ dy + [2^n t]^2|\Omega| \\ &\leqq [2^{2n}+1]t\|f\|_1.\end{aligned}$$

Hence it follows from Lemma 3.1 that

$$|\{x : Tg(x) > t\}| \leqq M[t^{-1}\|g\|_2]^2 \leqq M\frac{\|f\|_1}{t}.$$

Next we are concerned with b. Set

$$b_j(x) = \begin{cases} b(x) & \text{for } x \in Q_j, \\ 0 & \text{otherwise.} \end{cases}$$

With the aid of (3.8), Lemma 3.2 gives

$$\int_{\mathbf{R}^n - \tilde{Q}_j} |Tb_j(x)|\, dx \leqq A\|b_j\|_1 \leqq 2A \int_{Q_j} |f(x)|\, dx.$$

In view of Lemma 3.1, the series $\sum_j b_j$ and $\sum_j Tb_j$ are seen to converge to b and Tb in $L^2(\mathbf{R}^n)$, respectively. Hence

$$|Tb(x)| \leqq \sum_j |Tb_j(x)| \qquad \text{a.e.}$$

and

$$\int_{\mathbf{R}^n - \tilde{\Omega}} |Tb(x)|\, dx \leqq 2A \int_{\Omega} |f(x)|\, dx,$$

where $\tilde{\Omega} = \bigcup_j \tilde{Q}_j$. Since $|\tilde{\Omega}| \leqq (2\sqrt{n})^n |\Omega|$, we finally obtain

$$|\{x : Tb(x) > t\}| \leqq |\tilde{\Omega}| + t^{-1} \int_{\mathbf{R}^n - \tilde{\Omega}} |Tb(x)|\, dx \leqq Mt^{-1}\|f\|_1.$$

Thus we finally establish

$$|\{x : Tf(x) > t\}| \leqq |\{x : Tg(x) > t/2\}| + |\{x : Tb(x) > t/2\}| \leqq Mt^{-1}\|f\|_1.$$

THEOREM 3.3. *Let k be a function in $L^1_{loc}(\mathbf{R}^n - \{0\})$ satisfying* (3.3) - (3.7). *Then there exists a constant $M > 0$ such that*

$$\|Tf\| \leqq M\|f\|_p \qquad \textit{for all } f \in L^p(\mathbf{R}^n).$$

PROOF. Since Tf is both of weak type (1,1) by Theorem 3.2 and of type (2,2) by Lemma 3.1, the interpolation techniques imply that in case $1 < p < 2$,

$$\|Tf\|_p \leqq A_p\|f\|_p \qquad \text{for all } f \in L^p(\mathbf{R}^n).$$

For $\varphi \in \mathcal{S}$, Lemma 3.3 implies that

$$\begin{aligned} T\varphi(x) &= \text{p.v.} \int k(x-y)\varphi(y)dy \\ &= \lim_{r \to 0} \int_{\mathbf{R}^n - B(x,r)} k(x-y)\varphi(y)dy \end{aligned}$$

exists for all x, and, moreover, we have for $\psi \in \mathcal{S}$,

$$\int [T\varphi(x)]\psi(x)\,dx = \int \varphi(y)[T\psi(y)]\,dy.$$

Hence it follows that for $2 < p < \infty$,

$$\begin{aligned} \|T\varphi\|_p &= \sup_{\|\psi\|_{p'}=1} \int [T\varphi(x)]\psi(x)\,dx \\ &\leqq \|\varphi\|_p \left(\sup_{\|\psi\|_{p'}=1} \|T\psi\|_{p'} \right) \leqq A_{p'}\|\varphi\|_p. \end{aligned}$$

Now the required result follows by approximating $f \in L^p(\mathbf{R}^n)$ by functions in $\mathcal{S}$.

THEOREM 3.4. *Let Ω be a homogeneous function on $\mathbf{S}$ of degree 0 possessing the cancellation property*

$$\int_{\mathbf{S}} \Omega(x)dS(x) = 0 \tag{3.9}$$

and the smoothness property

$$\int_0^1 \frac{\omega(r)}{r}\,dr < \infty, \qquad \omega(r) = \sup_{|x-y|<r, |x|=1, |y|=1} |\Omega(x) - \Omega(y)|. \tag{3.10}$$

For $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$, and $\varepsilon > 0$, set

$$T_\varepsilon f(x) = \int_{\mathbf{R}^n - B(x,\varepsilon)} K(x-y)f(y)dy, \qquad K(y) = \frac{\Omega(y/|y|)}{|y|^n}.$$

Then :

(i) $\|T_\varepsilon f\|_p \leqq A_p\|f\|_p$ *for some positive constant A_p independent of f and ε.*

(ii) $T_\varepsilon f \to Tf$ *in $L^p(\mathbf{R}^n)$ as $\varepsilon \to 0$ and*

$$\|Tf\|_p \leqq A_p\|f\|_p.$$

Before proving this, we prepare several lemmas.

LEMMA 3.4. *Set $K(x) = |x|^{-n}\Omega(x/|x|)$. Then*

$$\int_{\{x:|x|>2|y|\}} |K(x-y) - K(x)|dx < \infty.$$

PROOF. Write

$$K(x-y) - K(x) = \frac{\Omega(x-y) - \Omega(x)}{|x-y|^n} + \Omega(x)\left(\frac{1}{|x-y|^n} - \frac{1}{|x|^n} \right).$$

We have

$$\begin{aligned}\int_{\{|x|>2|y|\}} \left|\frac{1}{|x-y|^n} - \frac{1}{|x|^n}\right| dx &\leqq M|y| \int_{\{|x|>2|y|\}} |x|^{-n-1}\, dx \\ &= M \int_{\{|x|>2\}} |x|^{-n-1}\, dx < \infty.\end{aligned}$$

Since $|(x-y)/|x-y| - x/|x|| \leqq A|y|/|x|$ when $|x| > 2|y|$, we see that

$$\begin{aligned}\int_{\{|x|>2|y|\}} \left|\frac{\Omega(x-y)-\Omega(x)}{|x-y|^n}\right| dx &\leqq M \int_{\{|x|>2|y|\}} \omega(A|y|/|x|)|x|^{-n}\, dx \\ &= M \int_0^{A/2} \frac{\omega(r)}{r}\, dr < \infty.\end{aligned}$$

Lemma 3.5. *The Fourier transform of $K(x)$ is a homogeneous function of degree* 0, *which is explicitly given by*

$$m(y) = \int_{\mathbf{S}} [(-\pi i/2)\mathrm{sgn}(y\cdot x) + \log(1/|y\cdot x|)]\Omega(x)dS(x), \qquad |y| = 1.$$

Proof. Setting

$$K_{r,R}(x) = \begin{cases} K(x) & \text{if } r < |x| < R, \\ 0 & \text{otherwise,} \end{cases}$$

we have only to show that

(i) $\sup |\mathcal{F}(K_{r,R})(y)| < A$ with a positive constant A independent of r and R.

(ii) $m(y) = \lim_{r\to 0, R\to\infty} \mathcal{F}(K_{r,R})(y)$ for $y \neq 0$.

Using polar coordinates, we see that

$$\mathcal{F}(K_{r,R})(y) = \int_{\mathbf{S}} \left(\int_r^R e^{-2\pi i s t y'\cdot x'} t^{-1} dt\right) \Omega(x')dS(x'),$$

where $x = tx'$ for $t = |x|$ and $y = sy'$ for $s = |y|$. By (3.3),

$$\mathcal{F}(K_{r,R})(y) = \int_{\mathbf{S}} I_{r,R}(y,x')\Omega(x')dS(x'),$$

with

$$I_{r,R}(y,x') = \int_r^R [e^{-2\pi i s t y'\cdot x'} - \cos(2\pi st)]t^{-1}dt.$$

The imaginary part of $I_{r,R}$ converges to

$$-\left(\int_0^\infty \frac{\sin t}{t}\, dt\right) \mathrm{sgn}(y'\cdot x') = (-\pi/2)\mathrm{sgn}(y'\cdot x')$$

as $r \to 0$ and $R \to \infty$. On the other hand, the real part of $I_{r,R}$ is equal to

$$\begin{aligned}\int_r^R [\cos(\lambda t) - \cos(\mu t)]t^{-1}dt &= \int_{\lambda r}^{\lambda R} \frac{\cos t}{t}\,dt - \int_{\mu r}^{\mu R} \frac{\cos t}{t}\,dt \\ &= \int_{\lambda r}^{\mu r} \frac{\cos t}{t}\,dt - \int_{\lambda R}^{\mu R} \frac{\cos t}{t}\,dt,\end{aligned}$$

which converges to

$$(\cos 0)\log(\mu/\lambda) = \log(1/|y' \cdot x'|)$$

as $r \to 0$ and $R \to \infty$. Moreover, we see that

$$|\mathcal{F}(K_{r,R})(y)| \leqq M \int_{\mathbf{S}} [1 + \log(1/|y' \cdot x'|)]|\Omega(x')|dS(x').$$

Now (i) and (ii) follow.

Now Theorem 3.4 follows from Theorem 3.3 together with Lemma 3.5.

THEOREM 3.5. *Let Ω be a homogeneous function on $\mathbf{S}$ of degree 0 possessing the cancellation property* (3.9) *and the smoothness property* (3.10). *For $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$, define*

$$T^* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|.$$

Then :

(i) $\lim_{\varepsilon \to 0} T_\varepsilon f(x)$ *exists for almost every x.*

(ii) *If $f \in L^1(\mathbf{R}^n)$, then $T^* f$ is of weak type $(1,1)$.*

(iii) *If $1 < p < \infty$, then*

$$\|T^* f\|_p \leqq A_p \|f\|_p.$$

Before proving this, we prepare the following result.

LEMMA 3.6. *For any $x \in \mathbf{R}^n$,*

$$T^* f(x) \leqq M(Tf)(x) + A\, Mf(x),$$

where M denotes the maximal operator.

PROOF. Let ψ be a nonnegative, nonincreasing and radial function on $\mathbf{R}^n$ such that

$$A = \int \psi(y)\, dy < \infty.$$

We first show that

$$\psi * f(x) \leqq AMf(x) \tag{3.11}$$

for nonnegative measurable function f on $\mathbf{R}^n$. In fact, letting

$$F(r) = \int_{B(x,r)} f(y)dy,$$

we see that

$$\begin{aligned} \psi * f(x) &= \int_0^\infty F(r)\, d(-\psi(r)) \\ &\leqq \int_0^\infty [|B(x,r)|Mf(x)]\, d(-\psi(r)) \\ &= [\sigma_n Mf(x)] \int_0^\infty r^n\, d(-\psi(r)) \\ &= n[\sigma_n Mf(x)] \int_0^\infty \psi(r) r^{n-1} dr = A\ Mf(x). \end{aligned}$$

Let φ be a nonnegative and radial function in $C_0^\infty(\mathbf{R}^n)$ such that $\varphi = 0$ outside the unit ball $\mathbf{B}$ and

$$\int \varphi(y)\ dy = 1.$$

Setting

$$K_\varepsilon(x) = \begin{cases} K(x) & \text{for } |x| > \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

we consider the function

$$\Phi(x) = \varphi * K(x) - K_1(x).$$

If $|x| \leqq 1$, then

$$\varphi * K(x) = \lim_{\varepsilon \to 0} \int_{\mathbf{R}^n - B(0,\varepsilon)} K(y)[\varphi(x-y) - \varphi(x)]dy,$$

so that Φ is bounded on $\mathbf{B}$ by the smoothness of φ. If $1 < |x| \leqq 2$, then

$$\Phi(x) = \lim_{\varepsilon \to 0} \int_{\mathbf{B}} [K(x-y) - K(x)]\varphi(y)dy,$$

so that Φ is also bounded there. If $|x| > 2$, then

$$|\Phi(x)| = \left| \int_{\mathbf{B}} [K(x-y) - K(x)]\varphi(y)dy \right|$$

so that

$$|\Phi(x)| \leqq A\omega(c/|x|)|x|^{-n}.$$

Since $\int_0^1 \omega(t)/t\ dt < \infty$, we see that $\psi(x) = \sup_{\{y:|y|>|x|\}} |\Phi(y)|$ is integrable. For $\varepsilon > 0$, note that

$$\Phi_\varepsilon = K * \varphi_\varepsilon - K_\varepsilon$$

with $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Hence it follows that

$$T_\varepsilon f = (Tf) * \varphi_\varepsilon - f * \Phi_\varepsilon,$$

which together with (3.11) proves the required inequality.

Proof of Theorem 3.5. Part (iii) follows from Lemma 3.6 and the L^p boundedness of maximal functions. To prove (ii), decompose $f \in L^1(\mathbf{R}^n)$ as $g + b$ in the proof of Theorem 3.2. We are only concerned with T^*b because of (iii). We show that for $x \in \tilde{F} \equiv \mathbf{R}^n - \bigcup_j \tilde{Q}_j$,

$$T^*b(x) \leqq \sum_j \int_{Q_j} |K(x-y) - K(x-y_j)||b(y)|dy + A\ Mb(x), \tag{3.12}$$

where y_j is the center of the cube Q_j. For this purpose, note first that

$$T_\varepsilon b(x) = \sum_j \int_{Q_j} K_\varepsilon(x-y)b(y)dy.$$

If $|x-y| > \varepsilon$ on Q_j, then

$$\int_{Q_j} K_\varepsilon(x-y)b(y)dy = \int_{Q_j} [K(x-y) - K(x-y_j)]b(y)dy.$$

Since $x \in \mathbf{R}^n - \tilde{Q}_j$, we have in general

$$\left| \int_{Q_j} K_\varepsilon(x-y)b(y)dy \right| \leqq AMb(x),$$

so that (3.12) follows. Hence we obtain

$$\begin{aligned} &\int_{\tilde{F}} \left(\sum_j \int_{Q_j} |K(x-y) - K(x-y_j)||b(y)|dy \right) dx \\ \leqq\ & \sum_j \int_{\mathbf{R}^n - \tilde{Q}_j} \left(\int_{Q_j} |K(x-y) - K(x-y_j)||b(y)|dy \right) dx \\ =\ & \sum_j \int_{Q_j} \left(\int_{\mathbf{R}^n - \tilde{Q}_j} |K(x-y) - K(x-y_j)|dx \right) |b(y)|dy \leqq AB\|b\|_1. \end{aligned}$$

Thus we find

$$\begin{aligned} |\{x : T^*b(x) > t\}| &\leqq t^{-1} \int_{\tilde{F}} T^*b(x)dx + \sum_j |\tilde{Q}_j| \\ &\leqq At^{-1}\|b\|_1 + Ct^{-1}\|f\|_1 \leqq At^{-1}\|f\|_1, \end{aligned}$$

which proves (ii).

Finally we show (i). Consider

$$I_f(x) = \left|\limsup_{\varepsilon\to 0}\ T_\varepsilon f(x) - \liminf_{\varepsilon\to 0}\ T_\varepsilon f(x)\right|.$$

If $f \in C_0^\infty(\mathbf{R}^n)$, then $I_f(x) = 0$ for all x. Moreover we see that

$$I_f(x) \leqq 2T^* f(x).$$

Thus, if we write $f = f_1 + f_2$, where $f_1 \in C_0^\infty(\mathbf{R}^n)$ and $\|f_2\|_p < \delta$, then $I_{f_1}(x) = 0$ for all x and

$$|\{x : I_f(x) > t\}| = |\{x : I_{f_2}(x) > t\}| \leqq 2A_p t^{-p}\|f_2\|_p \leqq 2A_p t^{-p}\delta.$$

Since δ can be chosen arbitrarily small, it follows that $|\{x : I_f(x) > t\}| = 0$, which implies that $I_f = 0$ a.e. on $\mathbf{R}^n$.

6.4 Applications of singular integral theory

For a multi-index λ and a positive integer ℓ, set

$$\kappa(x) = \kappa_{\lambda,\ell}(x) = \frac{x^\lambda}{|x|^\ell}.$$

If μ is a multi-index with length m, then $D^\mu\kappa$ is of the form

$$D^\mu\kappa(x) = \sum_{j=0}^{m} \frac{a_j(x)}{|x|^{\ell+2j}}, \tag{4.1}$$

where $a_j(x)$ is a homogeneous polynomial of degree $|\lambda|-m+2j$. In case $m = |\lambda|-\ell+n$, we write

$$\Omega(x/|x|) = \Omega_{\lambda,\ell,\mu}(x/|x|) = \sum_{j=0}^{m} \frac{a_j(x)}{|x|^{\ell+2j-n}}$$

and

$$K(x) = k_{\lambda,\ell,\mu}(x) = \frac{1}{|x|^n}\Omega\left(\frac{x}{|x|}\right).$$

It is seen from (4.1) that Ω is a homogeneous function of degree 0. Further Ω satisfies the Lipschitz condition on the unit sphere $\mathbf{S}$; that is, if $|x| = |y| = 1$, then

$$|\Omega(x) - \Omega(y)| \leqq M|x - y|.$$

LEMMA 4.1. *Ω has the cancellation property :*

$$\int_{\mathbf{S}} \Omega(x)dS(x) = 0. \tag{4.2}$$

PROOF. For a multi-index μ, we observe

$$\int_{\mathbf{S}} x^\mu dS(x) = \frac{2\prod_{j=1}^{n}\left(\frac{1+(-1)^{\mu_j}}{2}\right)\Gamma\left(\frac{\mu_j+1}{2}\right)}{\Gamma\left(\frac{n+|\mu|}{2}\right)}. \tag{4.3}$$

If $|\lambda| - \ell + n = |\mu| = 1$, then

$$D^\mu \kappa(x) = \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \frac{x^{\lambda-\mu}}{|x|^\ell} - \ell \frac{x^{\lambda+\mu}}{|x|^{\ell+2}},$$

where

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{cases} \prod_{j=1}^{n} \begin{pmatrix} \lambda_j \\ \mu_j \end{pmatrix} & \text{when } \lambda \geqq \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Here we wrote $\lambda \geqq \mu$ when $\lambda_j \geqq \mu_j$ for all j. Hence, in this case, we have by (4.3)

$$\int_{\mathbf{S}} \Omega_{\lambda,\ell,\mu}(x) dS(x) = 0. \tag{4.4}$$

Next assume that (4.4) holds for $m = |\lambda| - \ell + n = |\mu|$. Our aim is to show that (4.4) holds for $m+1 = |\lambda'| - \ell' + n = |\mu'|$. Write $\mu' = \mu + \nu$ with $|\mu| = m$ and $|\nu| = 1$. Then

$$D^{\mu'}\left(\frac{x^{\lambda'}}{|x|^{\ell'}}\right) = \begin{pmatrix} \lambda' \\ \nu \end{pmatrix} D^\mu \left(\frac{x^{\lambda'-\nu}}{|x|^{\ell'}}\right) - \ell' D^\mu \left(\frac{x^{\lambda'+\nu}}{|x|^{\ell'+2}}\right).$$

Since $|\lambda' - \nu| - \ell' + n = m$ if $\lambda' \geqq \nu$ and $|\lambda' + \nu| - (\ell' + 2) + n = m$, it follows from assumption on induction that

$$\int_{\mathbf{S}} D^\mu \left(\frac{x^{\lambda'-\nu}}{|x|^{\ell'}}\right) dS(x) = 0$$

and

$$\int_{\mathbf{S}} D^\mu \left(\frac{x^{\lambda'+\nu}}{|x|^{\ell'+2}}\right) dS(x) = 0,$$

so that

$$\int_{\mathbf{S}} D^{\mu'} \left(\frac{x^{\lambda'}}{|x|^{\ell'}}\right) dS(x) = 0.$$

Thus Lemma 4.1 is proved by induction.

Now Theorem 3.4 gives readily the following result.

THEOREM 4.1. *If $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$, then*

$$T_{\varepsilon,\mu} f(x) = \int_{\mathbf{R}^n - B(x,\varepsilon)} (D^\mu \kappa)(x-y) f(y) dy$$

converges to

$$T_\mu f(x) = \text{p.v.} \int (D^\mu \kappa)(x-y) f(y) dy$$

in $L^p(\mathbf{R}^n)$ as $\varepsilon \to 0$.

In particular,

$$R_j f(x) = \text{p.v.} c_n \int \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy, \qquad j = 1, 2, ..., n,$$

are called Riesz transforms, where

$$c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}.$$

THEOREM 4.2. *For $f \in L^2(\mathbf{R}^n)$,*

$$\mathcal{F}(R_j f)(y) = -i \frac{y_j}{|y|} \hat{f}(y), \qquad R_j(x) = c_n \frac{x_j}{|x|^{n+1}}.$$

PROOF. In view of Lemma 3.5,

$$m_j(y) \equiv \hat{R}_j(y) = c_n \int_{\mathbf{S}} x_j [(-\pi i/2) \text{sgn}(y \cdot x)] dS(x).$$

Here it suffices to note the identity

$$L(h) \equiv c_n (\pi/2) \int_{\mathbf{S}} x \cdot h \ \text{sgn}(y \cdot x) \ dS(x) = \frac{y \cdot h}{|y|};$$

in fact, L is linear, $|L(h)| \leqq |h|$ and $L(y) = |y|$.

Now consider the functions

$$\kappa_\varepsilon(x) = \kappa_{\lambda,\ell,\varepsilon}(x) = \frac{x^\lambda}{(|x| + \varepsilon)^\ell}$$

and

$$K_{\varepsilon,\mu}(x) = \begin{cases} (D^\mu \kappa_\varepsilon)(x) & \text{when } |x| > \varepsilon, \\ 0 & \text{when } |x| \leqq \varepsilon \end{cases}$$

for $\varepsilon > 0$, multi-indices λ, μ and a positive integer ℓ. Let f be a function in $L^p(\mathbf{R}^n)$ such that

$$\int (1 + |y|)^{m-n} |f(y)| dy < \infty \tag{4.5}$$

and set

$$\kappa f(x) = \int \kappa(x-y)f(y)dy = \int \frac{(x-y)^\lambda}{|x-y|^\ell} f(y)dy,$$

$$\kappa_\varepsilon f(x) = \int \kappa_\varepsilon(x-y)f(y)dy.$$

Then (4.5) implies that $\kappa|f|$ is locally integrable on $\mathbf{R}^n$.

LEMMA 4.2. *If $|\mu| = m$ and $|\lambda| - \ell + n = m$, then $D^\mu(\kappa_\varepsilon f)$ converges in $L^p(\mathbf{R}^n)$ as $\varepsilon \to 0$, and*

$$\|D^\mu(\kappa_\varepsilon f)\|_p \leqq A\|f\|_p.$$

PROOF. We write

$$D^\mu(\kappa_\varepsilon f)(x) - T_{\varepsilon,\mu} f(x) = \varepsilon^{-n} \int \theta_\mu(\varepsilon^{-1}(x-y))f(y)dy,$$

where $\theta_\mu(x) = D^\mu \kappa_1 - \chi_{\mathbf{R}^n - B(0,1)} D^\mu \kappa$. Note that $\theta_\mu(x) = O(|x|^{-n-1})$ as $|x| \to \infty$, which implies that $\theta_\mu \in L^1(\mathbf{R}^n)$. Hence it follows that

$$\|D^\mu \kappa_\varepsilon f(x) - T_{\varepsilon,\mu} f(x) - A_\mu f\|_p \to 0 \qquad \text{as } \varepsilon \to 0$$

with $A_\mu = \int \theta_\mu(x)dx$, and Lemma 4.2 is proved with the aid of Theorem 4.1.

THEOREM 4.3. *If $|\mu| = m$ and $|\lambda| - \ell + n = m$, then $D^\mu(\kappa f) \in L^p(\mathbf{R}^n)$ and*

$$D^\mu(\kappa f) = T_\mu f + A_\mu f. \tag{4.6}$$

PROOF. First note that $\kappa_\varepsilon f$ converges to κf in $L^1_{loc}(\mathbf{R}^n)$. Consequently we have for $\psi \in C_0^\infty(\mathbf{R}^n)$ and $|\mu| = m$,

$$\int (\kappa f) D^\mu \psi dx = \lim_{\varepsilon \to 0} \int (\kappa_\varepsilon f) D^\mu \psi dx.$$

Lemma 4.2 implies that

$$\left| \int (\kappa_\varepsilon f) D^\mu \psi dx \right| \leqq M \|D^\mu(\kappa_\varepsilon f)\|_p \|\psi\|_{p'} \leqq M\|f\|_p \|\psi\|_{p'}.$$

Hence it follows that

$$\|D^\mu(\kappa f)\|_p \leqq M\|f\|_p.$$

Moreover, in view of Theorem 4.1 and the proof of Lemma 4.2, we see that

$$D^\mu(\kappa f) = T_\mu f + A_\mu f,$$

as required.

COROLLARY 4.1. *If* $|\lambda| = m$ *and* $\ell \leqq m - n/p < \ell + 1$, *then*

$$\|D^\mu(k_{\lambda,\ell}f)\|_p \leqq M\|f\|_p \qquad \textit{for any multi-index } \mu \textit{ with length } m.$$

In fact, it suffices to see that $D^\mu(k_{\lambda,\ell}f) = D^\mu(k_\lambda f)$ for any multi-index μ with length m.

REMARK 4.1. Consider $\kappa(x) = |x|^{1-n}$ for $n \geqq 2$. Then we have for $f \in L^p(\mathbf{R}^n)$

$$(\partial/\partial x_j)U_1 f = [(1-n)/c_n]R_j f$$

because $A_j = \int \theta_j(x)dx = 0$ with $\theta_j(x) = (\partial/\partial x_j)\kappa_1(x) - \chi_{\mathbf{R}^n - B(0,1)}(x)(\partial/\partial x_j)\kappa(x)$.

REMARK 4.2. Consider $\kappa(x) = |x|^{2-n}$ for $n \geqq 3$. If $f \in L^p(\mathbf{R}^n)$, then

$$(\partial/\partial x_j)^2 U_2 f = T_\mu f - n^{-1}(n-2)\omega_n f$$

for $D^\mu = (\partial/\partial x_j)^2$; thus, $\Delta U_2 f = -(n-2)\omega_n f$.

6.5 Partial differentiability

In this section we discuss pointwise differentiability for Beppo Levi functions. In view of the integral representation, we are concerned with the functions of the form

$$k_{\lambda,\ell}f(x) = \int k_{\lambda,\ell}(x,y)f(y)dy$$

where $|\lambda| = m$, $\ell \leqq m - n/p < \ell + 1$ and $f \in L^p(\mathbf{R}^n)$. Corollary 4.1 implies that

$$k_{\lambda,\ell}f \in BL_m(L^p(\mathbf{R}^n)).$$

Note further that for any ball B,

$$u_1(x) = \int_B k_{\lambda,\ell}(x,y)f(y)dy = \int_B k_\lambda(x-y)f(y)dy + \text{a polynomial}$$

and

$$u_2(x) = \int_{\mathbf{R}^n - B} k_{\lambda,\ell}(x,y)f(y)dy$$

is infinitely differentiable inside B. Hence

$$\int |k_{\lambda,\ell}(x,y)|\ |f(y)|dy < \infty$$

if and only if

(5.1) $$\int_{B(x,1)} |x-y|^{m-n}|f(y)|dy < \infty.$$

LEMMA 5.1. *Set*

$$E = \left\{ x : \int |k_{\lambda,\ell}(x,y)|\, |f(y)| dy = \infty \right\}$$

for $|\mu| = m$, $\ell \leqq m - n/p < \ell + 1$ *and* $f \in L^p(\mathbf{R}^n)$. *Then* $C_{m,p}(E) = 0$.

LEMMA 5.2. *For each positive integer* j, *define*

$$v_j(x) = \int_B \frac{(x-y)^\lambda}{[|x-y| + (1/j)]^n} f(y) dy;$$

set

$$v_\infty(x) = \int_B \frac{(x-y)^\lambda}{|x-y|^n} f(y) dy.$$

Then v_j *is infinitely differentiable on* $\mathbf{R}^n$ *and* $v_j(x) \to v_\infty(x)$ *as* $j \to \infty$ *for every* $x \in B - E$.

In fact, $v_j(x) \to v_\infty(x)$ for every $x \in B$ such that (5.2) holds.

Denote by E^* the projection of E to the hyperplane $\mathbf{H}$. Then, in view of Theorem 5.1 in Chapter 5,

$$C_{m,p}(E^*) = 0. \tag{5.2}$$

This implies that E^* has Hausdorff dimension at most $n - mp$ on account of Corollary 2.1 in Chapter 5; in particular,

$$H_{n-1}(E^*) = 0. \tag{5.3}$$

A function u is called ACL if u is absolutely continuous along almost every lines parallel to the coordinate axes.

LEMMA 5.3. *Let* u *be a Borel function on a cube* C *with sides parallel to the coordinate axes. If* u *is ACL on* C, *then the first partial derivatives are finite a.e. on* C *and measurable on* C.

PROOF. For each positive integer j, consider the function

$$\frac{u(x_1 + (1/j), x') - u(x_1, x')}{(1/j)},$$

which is Borel measurable on C. If we set the right upper Dini derivative with respect to x_1

$$\overline{D_1^+} u(x_1, x') = \limsup_{j \to \infty} \frac{u(x_1 + (1/j), x') - u(x_1, x')}{(1/j)},$$

then it is Borel measurable on C. Since u is ACL on C, we see that $D_1^+ u(x)$ is equal to the first partial derivative of u with respect to x_1 for a.e. $x \in C$. Thus the first partial derivatives of u are all measurable on C.

COROLLARY 5.1. *Let u be a measurable function on an open set G. If u is ACL on G, then the first partial derivatives are finite a.e. on G and measurable on G.*

LEMMA 5.4. *Let $|\lambda| = 1$. Then v_∞ is ACL on $\mathbf{R}^n$.*

PROOF. By Theorem 4.1, $\|D^\mu(v_j - v_\infty)\|_p \to 0$ as $j \to \infty$ for $|\mu| = 1$. Hence there exists a sequence $\{j(i)\}$ such that

$$\lim_{i\to\infty} \int_{\mathbf{R}} |D^\mu(v_{j(i)} - v_\infty)(x_1, x')|^p dx_1 = 0 \tag{5.4}$$

for almost every $x' \in \mathbf{R}^{n-1}$. Further, if $(0, x') \notin E^*$, then

$$\lim_{i\to\infty} v_{j(i)}(x_1, x') = v_\infty(x_1, x') \qquad \text{for every } x_1 \in \mathbf{R}. \tag{5.5}$$

Hence if both (5.4) and (5.5) hold for $x' \in \mathbf{R}^{n-1}$, then we see that

$$\begin{aligned} v_\infty(b, x') - v_\infty(a, x') &= \lim_{i\to\infty} \{v_{j(i)}(b, x') - v_{j(i)}(a, x')\} \\ &= \lim_{i\to\infty} \int_a^b [(\partial/\partial x_1)v_{j(i)}](t, x')dt \\ &= \int_a^b [(\partial/\partial x_1)v_\infty](t, x')dt \end{aligned}$$

for every a, $b \in \mathbf{R}$, which implies that $v_\infty(\cdot, x')$ is absolutely continuous on $\mathbf{R}$. Noting (5.3), we see that v_∞ is ACL on $\mathbf{R}^n$.

THEOREM 5.1. *Let $u \in BL_1(L^p(\mathbf{R}^n))$. If $1 < p < \infty$, then u is equal almost everywhere to an ACL function on $\mathbf{R}^n$.*

For a locally integrable function f, we recall the definition of maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

and that

$$\|Mf\|_p \leqq A\|f\|_p$$

with a positive constant A independent of f, when $1 < p < \infty$.

LEMMA 5.5. *Let $|\lambda| > 1$. Then*

$$D^\mu v_\infty(x) = \int_B D^\mu \left(\frac{(x-y)^\lambda}{|x-y|^n} \right) f(y)dy$$

for any μ, $|\mu| = 1$, and any $x \in \mathbf{R}^n - A$ with $C_{m-1,p}(A) = 0$.

PROOF. Consider the maximal function

$$\overline{f}(x) = \sup_{h \in \mathbf{R}, h \neq 0} \frac{1}{h} \int_0^h |f(y_1 + t, y')| dt.$$

Then $\overline{f} \in L^p(\mathbf{R}^n)$. Define the set

$$A_1 = \left\{ x : \int_{B(x,1)} |x - y|^{m-1-n} \overline{f}(y) dy = \infty \right\}.$$

Then $C_{m-1,p}(A_1) = 0$. If $x \notin A_1$, then we have by Lebesgue's dominated convergence theorem

$$\lim_{h \to 0} \int_B \frac{\partial k_\lambda}{\partial x_1}(x - y) \left(\frac{1}{h} \int_0^h f(y_1 + t, y') dt \right) dy = \int_B \frac{\partial k_\lambda}{\partial x_1}(x - y) f(y) dy.$$

Moreover, setting $f_B = f \chi_B$, we have

$$\begin{aligned} & \int \frac{\partial k_\lambda}{\partial x_1}(x - y) \left(\frac{1}{h} \int_0^h f_B(y_1 + t, y') dt \right) dy \\ = \;& \frac{1}{h} \int \frac{\partial k_\lambda}{\partial x_1}(x - y) \left(\int_{y_1}^{y_1 + h} f_B(s, y') ds \right) dy \\ = \;& \frac{1}{h} \int \left(- \int_{s-h}^{s} \frac{\partial k_\lambda}{\partial y_1}(x - y) dy_1 \right) f_B(s, y') ds dy' \\ = \;& \frac{1}{h} \int \{ k_\lambda(x_1 + h - s, x' - y') - k_\lambda(x_1 - s, x' - y') \} f_B(s, y') ds dy'. \end{aligned}$$

Thus we see that v_∞ is partially differentiable at x with respect to x_1.

COROLLARY 5.2. *Let $m = |\lambda| > 1$. Then the first order partial derivatives of $K_{\lambda,\ell} f$ exist $(m-1,p)$-q.e. on $\mathbf{R}^n$ and are ACL on $\mathbf{R}^n$.*

THEOREM 5.2. *Let $u \in BL_m(L^p(\mathbf{R}^n))$. If $1 < p < \infty$, then u is equal almost everywhere to a function which, together with the partial derivatives of order at most $m - 1$, is ACL on $\mathbf{R}^n$.*

6.6 Beppo Levi spaces

Let us begin with the following result.

LEMMA 6.1. *Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a cube in $\mathbf{R}^n$. If $u_j \in C^1(I)$ satisfies*

$$D_i u_j = D_j u_i$$

on I for any i and j, $1 \leqq i,\ j \leqq n$, then there exists $u \in C^2(I)$ such that

$$D_j u = u_j$$

for all j, where $D_i = \partial/\partial x_i$.

In fact, the function

$$\begin{aligned} u(x_1, ..., x_n) \ &= \ \int_{a_1}^{x_1} u_1(t_1, a_2, ..., a_n)dt_1 \\ &+ \int_{a_2}^{x_2} u_2(x_1, t_2, a_3, ..., a_n)dt_2 \\ &\vdots \\ &+ \int_{a_n}^{x_n} u_n(x_1, x_2, ..., x_{n-1}, t_n)dt_n. \end{aligned}$$

satisfies

$$D_j u = u_j.$$

Corollary 6.1. *Let G be a domain in $\mathbf{R}^n$. If $u_j \in C^1(G)$ satisfies*

$$D_i u_j = D_j u_i$$

on I for any i and j, $1 \leqq i,\ j \leqq n$, then there exists $u \in C^2(G)$ such that

$$D_j u = u_j \qquad \text{for all } j.$$

Lemma 6.2. *Let G be a domain in $\mathbf{R}^n$. If $u_\lambda \in C^m(G)$ satisfies*

$$D^\mu u_\lambda = D^{\mu'} u_{\lambda'} \tag{6.1}$$

on G, for any multi-indices μ, μ', λ and λ' such that $|\mu| = |\mu'| = 1$, $|\lambda| = |\lambda'| = m$ and $\mu + \lambda = \mu' + \lambda'$, then there exists $u \in C^{m+1}(G)$ such that

$$D^\lambda u = u_\lambda \tag{6.2}$$

on G, for any multi-index λ with length m.

Proof. We show this lemma by induction on m. First note from Corollary 6.1 that the present lemma is true for $m = 1$. Suppose this is true for m. For each λ with $|\lambda| = m$, we can find v_λ such that

$$D^\nu v_\lambda = u_{\lambda+\nu} \qquad \text{for any } |\nu| = 1,$$

on account of Corollary 6.1. Next, by assumption on induction, we can find u such that

$$D^\lambda u = v_\lambda \qquad \text{for any } |\lambda| = m.$$

It is easy to see that u satisfies (6.2) for $m+1$.

LEMMA 6.3. *Let G be a bounded open set in $\mathbf{R}^n$ and $f \in C^k(G)$. Then a solution $\Delta u = f$ in G belongs to the class $C^{k+1}(G)$.*

PROOF. Consider the function

$$h(x) = u(x) - c_n \int_G N(x-y) f(y) dy,$$

where $c_n = -(a_n \omega_n)^{-1}$ is chosen so that

$$\Delta h = \Delta u - f|_G$$

in the sense of distributions. Thus h is harmonic in G. For $x_0 \in G$, take $\varphi \in C_0^\infty(G)$ which equals 1 on a neighborhood of x_0. Then note that

$$\int_G N(x-y)[(1-\varphi(y))f(y)]dy$$

is harmonic in a neighborhood of x_0 and

$$D^\lambda \left(\int N(x-y)\varphi(y)f(y)dy \right) = \int N(x-y) D^\lambda[\varphi(y)f(y)]dy$$

for $|\lambda| = k$, which shows that the potential of f is in $C^{k+1}(G)$.

We say that a function $h \in C^\infty(G)$ is polyharmonic of order m in G if $\Delta^m h = 0$ on G.

THEOREM 6.1. *Let G be a domain in $\mathbf{R}^n$. If $u_\lambda \in L^p(G)$ satisfies (6.1) on G in the sense of distributions, for any multi-indices μ, μ', λ and λ' such that $|\mu| = |\mu'| = 1$, $|\lambda| = |\lambda'| = m$ and $\mu + \lambda = \mu' + \lambda'$, then there exists $u \in BL_m(L^p(G))$ which satisfies (6.2) on G in the sense of distributions, for any multi-index λ with length m.*

PROOF. Consider the function

$$v(x) = \sum_{|\lambda|=m} b_\lambda \int_G \tilde{k}_{\lambda,\ell}(x,y) u_\lambda(y) dy,$$

where $\ell \leqq m - n/p < \ell + 1$. We show below that

$$\Delta^m (D^\mu v - u_\mu) = 0 \qquad \text{on } G. \tag{6.3}$$

To show this, taking $\psi \in C_0^\infty(G)$ and $|\mu| = m$, we have as in the proof of Theorem 2.2

$$\int v(x) D^\mu(\Delta^m \psi(x)) dx = \sum_{|\lambda|=m} b_\lambda \int_G \left(\int \tilde{k}_{\lambda,\ell}(x,y) \Delta^m D^\mu \psi(x) dx \right) u_\lambda(y) dy$$

$$\begin{aligned}
&= \sum_{|\lambda|=m} b_\lambda \int_G \left((-1)^m b_m^{-1} D^\lambda D^\mu \psi(y)\right) u_\lambda(y)dy \\
&= (-1)^m \sum_{|\lambda|=m} m!/\lambda! \int_G D^\mu (D^\lambda \psi(y)) u_\lambda(y)dy \\
&= (-1)^m \sum_{|\lambda|=m} m!/\lambda! \int_G D^\lambda (D^\lambda \psi(y)) u_\mu(y)dy \\
&= (-1)^m \int_G u_\mu(y) \Delta^m \psi(y)dy,
\end{aligned}$$

which shows that (6.3) holds. Let $h_\mu = u_\mu - D^\mu v$, which is a polyharmonic function of order m on G by Lemma 6.3. In view of Lemma 6.2, there exists a function $h \in C^m(G)$ such that $D^\mu h = h_\mu$ on G for any multi-index μ with length m. Now we see that

$$u(x) = \sum_{|\lambda|=m} b_\lambda \int_G \tilde{k}_{\lambda,\ell}(x,y) u_\lambda(y)dy + h(x)$$

is the required function.

Corollary 6.2. *If $u \in BL_m(L^p(G))$, then*

$$u(x) = \sum_{|\lambda|=m} b_\lambda \int_G \tilde{k}_{\lambda,\ell}(x,y) D^\lambda u(y)dy + h(x) \tag{6.4}$$

for h polyharmonic of order m in G.

Remark 6.1. This corollary gives an extension of Riesz decomposition theorem to Beppo Levi functions, given for superharmonic functions on G (see Theorem 2.3 in Chapter 3).

Let G be an open set in $\mathbf{R}^n$. We write $u \in BL_m(L^p_{loc}(G))$ if $u \in BL_m(L^p(G'))$ for any open set G' with compact closure in G. The following is a consequence of Sobolev's theorem (Theorem 2.1 in Chapter 4).

Corollary 6.3. *If $u \in BL_m(L^p_{loc}(G))$, then $D^\lambda u \in L^p_{loc}(G)$ for any multi-index λ with $|\lambda| \leqq m$.*

If $D^\lambda u \in L^p(G)$ for any multi-index λ with $|\lambda| \leqq m$, then we write $u \in W^{m,p}(G)$, and say that $W^{m,p}(G)$ is a Sobolev space on G.

In view of Corollary 6.3, we have the following result.

Corollary 6.4. *If $u \in BL_m(L^p_{loc}(G))$, then $\psi u \in W^{m,p}(G)$ for all $\psi \in C_0^\infty(G)$.*

By Theorem 5.2 and (6.4), we have the following result.

Theorem 6.2. *Let $u \in BL_m(L^p_{loc}(G))$. If $1 < p < \infty$, then u is equal almost everywhere to a function which, together with the partial derivatives of order at most $m - 1$, is ACL on G.*

To consider the quasi-norm in $L^p_{loc}(G)$, let $\{G_j\}$ be a sequence of relatively compact open subsets of G such that $G_j \subsetneqq \overline{G}_j \subsetneqq G_{j+1}$ and $\bigcup_j G_j = G$. Define

$$A(u) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|u\|_{L^p(G_j)}}{1 + \|u\|_{L^p(G_j)}}.$$

Denote by $F = BL_m(L^p(G))$ with the quasi-norm

$$B(u) = A(u) + \left(\int_G |\nabla^m u(y)|^p dy \right)^{1/p}.$$

LEMMA 6.4. *Let $F^\bullet$ be the quotient space of F by the space of polynomials of degree at most $m-1$. Then $BL_m(L^p(G))^\bullet$ is a Banach space and $F^\bullet$ is a Fréchet space. Moreover, the mapping $u^\bullet \to u^\bullet$ from $BL_m(L^p(G))^\bullet$ to $F^\bullet$ is an isomorphism.*

PROOF. Let $\{u_\lambda\}$ be a sequence such that $D^\lambda u_j \to u_\lambda$ in $L^p(G)$ for all λ with length m. Since

$$D^\mu u_\lambda = D^{\lambda'} u_{\mu'}$$

whenever $|\lambda| = |\mu'| = m$ and $\mu + \lambda = \lambda' + \mu'$, by Theorem 6.1 there exists $u \in BL_m(L^p(G))$ for which

$$D^\lambda u = u_\lambda \qquad \text{for any multi-index } \lambda \text{ with length } m.$$

Thus $BL_m(L^p(G))^\bullet$ is a Banach space. Similarly, $F^\bullet$ is a Fréchet space.

The second assertion follows from closed graph theorem.

For $u \in BL_m(L^p(G))$, define a seminorm

$$|u|_{m,p} = \left(\sum_{|\lambda|=m} \|D^\lambda u\|_p^p \right)^{1/p};$$

for $u \in W^{m,p}(G)$, define a norm

$$\|u\|_{m,p} = \left(\sum_{|\lambda| \leqq m} \|D^\lambda u\|_p^p \right)^{1/p}.$$

THEOREM 6.3. *Let G be a domain in $\mathbf{R}^n$. If $\{u_j\}$ is a Cauchy sequence in $BL_m(L^p(G))$, then there exist $u \in BL_m(L^p(G))$ and a sequence $\{P_j\}$ of polynomials of degree at most $m-1$ for which $u_j + P_j \to u$ in $L^p_{loc}(G)$ and*

$$\lim_{j \to \infty} |u_j - u|_{m,p} = 0.$$

THEOREM 6.4. *If $u \in BL_m(L^p(G))$, then there exists a sequence $\{\psi_j\}$ in $C^\infty(G)$ for which*

$$\lim_{j\to\infty} |\psi_j - u|_{m,p} = 0.$$

PROOF. Let $\{a_j\}$ be a partition of unity in G, and set

$$u_j = a_j u.$$

Note here that $u_j \in W^{m,p}(\mathbf{R}^n)$ and $u = \sum_j u_j$. By considering a mollifier, we can find $\psi_{i,j} \in C_0^\infty(G)$ such that

$$\|\psi_{i,j} - u_j\|_{m,p} < i^{-1}2^{-j}.$$

Further we may assume that for any i, $S_{\psi_{i,j}}$, the support of $\psi_{i,j}$, intersects with the other ones at most N times. Now $\psi_i = \sum_j \psi_{i,j}$ is the required one.

COROLLARY 6.5. *If $u \in W^{m,p}(G)$, then there exists a sequence $\{\psi_j\}$ in $C^\infty(G)$ for which*

$$\lim_{j\to\infty} \|\psi_j - u\|_{m,p} = 0.$$

LEMMA 6.5. *If $u \in BL_m(L^p(\mathbf{R}^n))$, $1 < p < \infty$, then*

$$\lim_{j\to\infty} \|\nabla^m(u * \psi_j - u)\|_p{}^p = 0$$

for a sequence $\{\psi_j\}$ of mollifiers.

This is easy if one notes that

$$\nabla^m(u * \psi_j - u) = (\nabla^m u) * \psi_j - \nabla^m u.$$

In view of Theorem 1.3 in Chapter 6, $u \in BL_m(L^p(\mathbf{R}^n))$ is represented as

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int_{\mathbf{R}^n} k_{\lambda,\ell}(x,y)[D^\lambda u(y)]dy + P$$

for some polynomial P, where $\ell \leqq m - n/p < \ell + 1$.

Corollary 4.1 in Chapter 6 proves the following.

LEMMA 6.6. *If $u \in BL_m(L^p(\mathbf{R}^n))$, $1 < p < \infty$, then*

$$v_N(x) = \sum_{|\lambda|=m} a_\lambda \int_{B(0,N)} k_{\lambda,\ell}(x,y)[D^\lambda u(y)]dy$$

converges to u in $BL_m(L^p(\mathbf{R}^n))$ as $N \to \infty$.

LEMMA 6.7. *Let ψ be a function in $C_0^\infty(\mathbf{R}^n)$ which equals 1 on a neighborhood of the origin. If we set*

$$w_N(x) = \sum_{|\lambda|=m} a_\lambda \int_{B(0,N)} k_\lambda(x-y)[D^\lambda u(y)]dy,$$

then $\psi(j^{-1}x)w_N(x) \to w_N(x)$ in $BL_m(L^p(\mathbf{R}^n))$ as $j \to \infty$.

PROOF. Noting that $|\nabla^{m-i}k_\lambda(x)| \leqq M|x|^{m-n-(m-i)}$ for $2|x| > |y|$, we have

$$\begin{aligned}\|\nabla^m(\psi(j^{-1}x)w_N(x) - w_N(x))\|_p{}^p &\leqq M\sum_{i=1}^m j^{-np}\int |(\nabla^i\psi)(j^{-1}x)|^p dx \\ &\quad + M\int |\psi(j^{-1}x)-1|^p|\nabla^m w_N(x)|^p dx \\ &\leqq Mj^{-np+n} + M\int |\psi(j^{-1}x)-1|^p|\nabla^m w_N(x)|^p dx \\ &\to 0 \qquad \text{as } j \to \infty.\end{aligned}$$

Here note that

$$v_N(x) = w_N(x) + P_\ell$$

with a polynomial P_ℓ of degree at most ℓ. In view of Lemmas 6.6 and 6.7, any function in $BL_m(L^p(\mathbf{R}^n))$ can be approximated by functions in $BL_m(L^p(\mathbf{R}^n))$ with compact support. Now, by Lemma 6.5, we have the following result.

THEOREM 6.5. *For $u \in BL_m(L^p(\mathbf{R}^n))$, there exists a sequence $\{\varphi_j\}$ in $C_0^\infty(\mathbf{R}^n)$ which converges to u in $BL_m(L^p(\mathbf{R}^n))$.*

6.7 Continuity properties of BLD functions

We say that a function u is (m,p)-quasicontinuous on a bounded open set G if for any $\varepsilon > 0$ and any open set G' with compact closure in G, there exists an open set ω such that $C_{m,p}(\omega, G) < \varepsilon$ and $u|_{G'-\omega}$ is continuous.

LEMMA 7.1. *If u is (m,p)-quasicontinuous on G, then for any $\varepsilon > 0$ there exists an open set ω such that $C_{m,p}(\omega, G) < \varepsilon$ and $u|_{G-\omega}$ is continuous.*

For this purpose it suffices to consider an exhaustion of G.

We show that Riesz potentials of functions in L^p are quasicontinuous.

THEOREM 7.1. *Let $\kappa(x) = x^\lambda/|x|^\ell$ for a multi-index λ with length $m+\ell-n \geqq 0$. If $f \in L^p(\mathbf{R}^n)$ and $U_m|f| \not\equiv \infty$, then $U_\kappa f$ is (m,p)-quasicontinuous on $\mathbf{R}^n$.*

PROOF. For $R > 0$, write

$$\begin{aligned} U_\kappa f(x) &= \int_{B(0,R)} \kappa(x-y)f(y)dy + \int_{\mathbf{R}^n - B(0,R)} \kappa(x-y)f(y)dy \\ &= u_1(x) + u_2(x). \end{aligned}$$

It is easy to see that u_2 is continuous on $B(0,R)$. For $N > 0$, define

$$u_{1,N}(x) = \int_{B(0,R)} \kappa(x-y) f_N(y)dy,$$

where $f_N = \max(\min(f, N), -N)$. Consider

$$E_{k,N} = \left\{ x : \int_{B(0,R)} |x-y|^{m-n} |f(y) - f_N(y)| dy > 2^{-k} \right\}.$$

Then it follows that

$$C_{m,p}(E_{k,N}, B(0,R)) < 2^{-kp} \int_{B(0,R)} |f(y) - f_N(y)|^p dy,$$

so that, for $N = N(k)$,

$$C_{m,p}(E_{k,N(k)}, B(0,R)) < 2^{-kp}.$$

Now, letting

$$\omega_j = \bigcup_{k=j}^{\infty} E_{k,N(k)},$$

we see that

$$C_{m,p}(\omega_j, B(0,R)) \leqq \sum_{k=j}^{\infty} C_{m,p}(E_{k,N(k)}, B(0,R)) < \sum_{k=j}^{\infty} 2^{-kp}$$

and $u_{1,N(k)}$ converges to u_1 uniformly on $B(0,R) - \omega_j$. Consequently, since $u_{1,N}$ is continuous on $\mathbf{R}^n$, u_1 is continuous as a function on $B(0,R) - \omega_j$. Thus $U_\kappa f$ is continuous as a function on $B(0,R) - \omega_j$, and the present theorem is obtained.

As applications of integral representations and Theorem 7.1, we have the following result.

THEOREM 7.2. *For any $u \in BL_m(L^p_{loc}(G))$, there exists an (m,p)-quasicontinuous function on G which equals u a.e. on G.*

A function $u \in BL_m(L^p_{loc}(G))$ is called BLD if it is (m,p)-quasicontinuous on G. Theorems 7.1 and 7.2 in Chapter 5 give the following result.

THEOREM 7.3. *Let u be a BLD function in $BL_m(L^p_{loc}(G))$. For $k < m$, u is k times (m,p)-finely differentiable $(m-k,p)$-q.e. on G. Further, u is m times (m,p)-semifinely differentiable a.e. on G.*

Theorems 8.2 and 8.3 in Chapter 5 give the following result.

THEOREM 7.4. *Let u be a BLD function in $BL_m(L^p_{loc}(G))$. If $mp > n$, then u is continuous on G. Moreover, for $k < m$, u is k times differentiable $(m-k,p)$-q.e. on G and u is m times differentiable a.e. on G.*

For a function u and $h \in \mathbf{R}^n$, set

$$\triangle_h^1 u(x) = u(x+h) - u(x)$$

and define $\triangle_h^m u(x) = \triangle_h^1(\triangle_h^{m-1} u)(x)$ inductively. Note that

$$\triangle_h^m u(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} u(x+jh).$$

THEOREM 7.5. *Let u be a BLD function in $BL_m(L^p_{loc}(G))$. If $mp > n$, then*

$$|\triangle_h^m (x)|^p \leqq M|h|^{mp-n} \int_{B(x,m|h|)} |\nabla_m u(z)|^p dz \tag{7.1}$$

whenever $B(x, m|h|) \subseteqq G$.

PROOF. We show (7.1) only in case $m = 1$. For $B(x, |x-y|) \subseteqq G$, writing $r = |x-y|$, we have

$$\begin{aligned}
&\left| u(y) - \frac{1}{|B(x,r)|} \int_{B(x,r)} u(z)dz \right| \leqq \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y) - u(z)|dz \\
\leqq\ & \frac{1}{|B(x,r)|} \int_{B(x,r)} \left(|y-z| \int_0^1 |\nabla u(z + t(y-z))|dt \right) dz \\
\leqq\ & 2r \int_0^1 \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |\nabla u(z + t(y-z))|^p dz \right)^{1/p} dt \\
\leqq\ & 2r \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |\nabla u(w)|^p dw \right)^{1/p} \int_0^1 (1-t)^{-n/p} dt \\
\leqq\ & M \left(r^{p-n} \int_{B(x,r)} |\nabla u(w)|^p dw \right)^{1/p},
\end{aligned}$$

which proves (7.1).

6.8 Dirichlet space

In this section we consider the family of BLD functions with gradient in L^p. We know that L^p spaces is separable for $1 \leqq p < \infty$, that is, any $L^p(G)$ has a countable dense subset. Hence we have the following result by taking a diagonal sequence.

LEMMA 8.1. *Let $\{f_j\}$ be a sequence of functions in $L^p(G)$, $1 \leqq p < \infty$, which is bounded. Then there exists a subfamily $\{f_{j(k)}\}$ which is weakly convergent to a function $f \in L^p(G)$, that is,*

$$\lim_{k\to\infty} \int_G f_{j(k)} g dx = \int_G f g dx \qquad \text{for all } g \in L^{p'}(G);$$

in addition,

$$\liminf_{k\to\infty} \|f_{j(k)}\|_p \geqq \|f\|_p.$$

By an application of Clarkson's inequality, we have the following result.

LEMMA 8.2. *Let $\{f_j\}$ be a sequence of functions in $L^p(G)$, $1 \leqq p < \infty$, which is weakly convergent to a function $f \in L^p(G)$. If*

$$\lim_{j\to\infty} \|f_j\|_p = \|f\|_p,$$

then $\{f_j\}$ is convergent to f in $L^p(G)$.

THEOREM 8.1. *Let $u \in BL_1(L^p(G))$. Then :*

(1) *For any number N, $u \wedge N \in BL_1(L^p(G))$; in particular, $u^+ \in BL_1(L^p(G))$, $u^- \in BL_1(L^p(G))$ and $|u| \in BL_1(L^p(G))$. Moreover, the mapping : $u \to u \wedge N$ is continuous in $BL_1(L^p(G))$.*

(2) *The mapping $u \to u \wedge v$ is continuous in $BL_1(L^p(G))$ for fixed $v \in BL_1(L^p(G))$.*

PROOF. If u is ACL on G, then so is $u \wedge N$ for any number N and

$$\nabla(u \wedge N) = \begin{cases} \nabla u & \text{a.e. on } \{x \in G : u(x) < N\}, \\ 0 & \text{a.e. on } \{x \in G : u(x) \geqq N\}, \end{cases}$$

so that

$$\|\nabla(u \wedge N)\|_p \leqq \|\nabla u\|_p. \tag{8.1}$$

Hence $u \wedge N \in BL_1(L^p(G))$. Further note that

$$u^+ = (-u) \wedge 0 \in BL_1(L^p(G)),$$

$$u^- = -(u \wedge 0) \in BL_1(L^p(G))$$

and

$$|u| = u^+ + u^- \in BL_1(L^p(G)).$$

Let $\{u_j\}$ be a sequence of functions in $BL_1(L^p(G))$ which converges to u with respect to the seminorm $|\cdot|_{1,p}$. Since $|\nabla|v|| = |\nabla v|$ for any $v \in BL_1(L^p(G))$, we see that

$$\lim_{j\to\infty} \|\nabla|u_j|\|_p = \|\nabla|u|\|_p,$$

so that

$$\lim_{j\to\infty} \|\nabla|u_j| - \nabla|u|\|_p = 0.$$

Since $u \wedge N = [(u+N) - |u-N|]/2$, the mapping $u \to u \wedge N$ is continuous with respect to the seminorm $|\cdot|_{1,p}$.

Finally, if we note that $u \wedge v = u - (u-v)^+$, then (2) follows.

We sometimes say that a function $u \in BL_1(L^p(G))$ is p-precise if it is $(1,p)$-quasicontinuous on G. We say that a property holds p-a.e. on G if it holds for every $x \in G$ except those in a set with $C_{1,p}$-capacity zero.

The following is an easy consequence of Theorem 6.3.

PROPOSITION 8.1. *Let $\{u_j\}$ be a sequence of p-precise functions which converges to u in $BL_1(L^p(G))$. Then there exist a subsequence $\{u_{j(k)}\}$ and a sequence $\{c_k\}$ of numbers such that $u_{j(k)} + c_k$ converges p-a.e. on G to a p-precise function u^* which is equal to u a.e. on G.*

PROPOSITION 8.2. *For $u \in BL_1(L^p(G))$, there exists a p-precise function which is equal to u almost everywhere on G. Further, if u and v are p-precise functions on G such that $u = v$ a.e. on G, then $u = v$ p-a.e. on G.*

PROPOSITION 8.3. *Let u be a p-precise function on $\mathbf{R}^n$. If $u = 0$ p-a.e. outside a bounded open set G, then there exists a sequence $\{u_j\}$ in $C_0^\infty(G)$ which converges to u in $BL_1(L^p(\mathbf{R}^n))$.*

PROOF. We may assume that u is bounded, and hence, from the beginning, we may assume that

$$0 \leqq u \leqq M \qquad \text{on } \mathbf{R}^n.$$

Since G is bounded, take a bounded open set D for which $\overline{G} \subsetneqq D$. For any $\varepsilon > 0$, we can find an open set ω such that $C_{1,p}(\omega, D) < \varepsilon$ and $u|_{D-\omega}$ is continuous. Further we can find an open set ω' such that $C_{1,p}(\omega', D) < \varepsilon$ and $u = 0$ on $D - (G \cup \omega')$. Take a nonnegative function $f_\varepsilon \in L^p(D)$ such that $v_\varepsilon = U_1 f_\varepsilon \geqq 1$ on $\omega \cup \omega'$ and $\|f_\varepsilon\|_p < 2\varepsilon$. Since $u = 0$ on $\partial G - (\omega \cup \omega')$, we see that

$$u_\varepsilon \equiv \max(0, u - \varepsilon - Mv) = 0$$

on a neighborhood of ∂G. On the other hand, if $\varepsilon \to 0$, then $u_\varepsilon \to \max(0,u) = u$ in $BL_1(L^p(\mathbf{R}^n))$. Hence, considering a sequence $\{\psi_j\}$ of mollifiers, we infer that $u_\varepsilon * \psi_j \in C_0^\infty(G)$ for large j and $u_\varepsilon * \psi_j \to u_\varepsilon$ in $BL_1(L^p(\mathbf{R}^n))$ as $j \to \infty$.

We consider the capacity

$$C_p(E;G) = \inf \ \|\nabla u\|_p{}^p,$$

where the infimum is taken over all nonnegative p-precise functions u on $\mathbf{R}^n$ such that $u \geqq 1$ p-a.e. on E and $u = 0$ p-a.e. on $\mathbf{R}^n - G$.

THEOREM 8.2. (1) *$C_p(\cdot, G)$ is countably subadditive and nondecreasing.*

(2) *$C_p(E;G) = \inf \ \|\nabla u\|_p{}^p$, where the infimum is taken over all nonnegative p-precise functions u on $\mathbf{R}^n$ such that $u = 1$ p-a.e. on E and $u = 0$ p-a.e. on $\mathbf{R}^n - G$.*

(3) *If K is a compact set in a bounded open set G, then*

$$M^{-1}C_p(E;G) \leqq C_{1,p}(E;G) \leqq MC_p(E;G) \qquad \text{whenever } E \subseteqq K.$$

PROOF. Since (1) and (2) are easy, we show (3) only. By Sobolev's integral representation, if u is p-precise on $\mathbf{R}^n$ and $u = 0$ outside G, then

$$|u(x)| \leqq |c| \int \ |x-y|^{1-n}|\nabla u(y)|dy$$

for all $x \in \mathbf{R}^n$. Hence

$$C_{1,p}(E;G) \leqq |c|^p C_p(E;G).$$

To show the converse, let f be a nonnegative measurable function in $L^p(G)$ such that $U_1 f \geqq 1$ on E. By considering $\chi \in C_0^\infty(G)$ which equals 1 on a neighborhood of K, we have by Theorem 4.3 together with Sobolev's theorem,

$$C_p(E;G) \leqq \|\nabla(\chi(U_1 f))\|_p^p \leqq M\|\nabla(U_1 f)\|_p^p \leqq M\|f\|_p^p,$$

so that

$$C_p(E;G) \leqq MC_{1,p}(E;G).$$

PROPOSITION 8.4. (1) *If $p \geqq n$, then $C_p(E;\mathbf{R}^n) = 0$ for any set E.*

(2) *If $p \geqq n$ and $C_p(E;G) = 0$ for some bounded open set G, then $C_{1,p}(E) = 0$.*

(3) *If $p > n$ and G is a bounded open set, then $C_p(E;G) > 0$ for any nonempty set E.*

(4) *In case $1 < p < n$, $C_p(E;\mathbf{R}^n) = 0$ if and only if $C_{1,p}(E) = 0$.*

PROOF. We show only (1). For this purpose, by countable subadditivity we may assume that E is bounded. Take $N > 0$ so that $E \subseteqq B(0,N)$. In case $p = n$, consider the function

$$u(x) = \begin{cases} 1 & \text{on } B(0,N), \\ \dfrac{\log(N^2/|x|)}{\log N} & \text{on } B(0,N^2) - B(0,N), \\ 0 & \text{on } \mathbf{R}^n - B(0,N^2). \end{cases}$$

Then we have

$$C_p(E;\mathbf{R}^n) \leqq C_p(B(0,N);\mathbf{R}^n) \leqq \|\nabla u\|_p{}^p \leqq M(\log N)^{1-p},$$

which implies that $C_p(E;\mathbf{R}^n) = 0$. If $p > n$, then we take $u \in C_0^\infty(\mathbf{R}^n)$ such that $u = 1$ on $\mathbf{B}$ and consider $u_N(x) = u(N^{-1}x)$. Then

$$C_p(E;\mathbf{R}^n) \leqq \|\nabla u_N\|_p{}^p \leqq MN^{n-p},$$

which implies that $C_p(E;\mathbf{R}^n) = 0$.

We further define the capacity

$$C_p^*(E) = \inf\ \|u\|_{1,p}{}^p,$$

where the infimum is taken over all nonnegative p-precise functions u on $\mathbf{R}^n$ for which $u \geqq 1$ p-a.e. on E.

PROPOSITION 8.5. *Let $\{u_j\}$ be a sequence of p-precise functions on G which converges to a p-precise function u in $W^{1,p}(G)$. Then there exists a subsequence $\{u_{j(k)}\}$ which converges to u for every $x \in G - E$ with $C_p^*(E) = 0$.*

PROPOSITION 8.6. (1) *C_p^* is countably subadditive and nondecreasing.*

(2) $C_p(E;G) = \inf\ \|u\|_{1,p}{}^p$, *where the infimum is taken over all nonnegative p-precise functions u on $\mathbf{R}^n$ such that $u = 1$ p-a.e. on E.*

(3) *If K is a compact set in a bounded open set G, then*

$$M^{-1}C_p(E;G) \leqq C_p^*(E) \leqq MC_p(E;G) \quad \text{whenever } E \subseteqq K.$$

(4) $C_p^*(E) = 0$ *if and only if* $C_{1,p}(E) = 0$.

Since C_p^* is an outer capacity, we have the following.

PROPOSITION 8.7. *Let K be a compact subset of an open set G. Then $C_p^*(K) = 0$ if and only if there exists a sequence $\{\psi_j\}$ in $C_0^\infty(G)$ such that $\psi_j = 1$ on a neighborhood of K and $\|\psi_j\|_{1,p} \to 0$ as $j \to \infty$.*

PROPOSITION 8.8. *Let K be a compact subset of an open set G. Then $C_p^*(K) = 0$ if and only if $C_0^\infty(G - K)$ is dense in $C_0^\infty(G)$ with respect to the norm $\|\cdot\|_{1,p}$.*

PROOF. If $C_p^*(K) = 0$, then by Proposition 8.7 there exists a sequence $\{\psi_j\}$ in $C_0^\infty(G)$ such that $\psi_j = 1$ on a neighborhood of K and $\|\psi_j\|_{1,p} \to 0$ as $j \to \infty$. If $\varphi \in C_0^\infty(G)$, then $(1 - \psi_j)\varphi \in C_0^\infty(G - K)$ and

$$\|\psi_j\varphi\|_{1,p} \leqq M(\varphi)\|\psi_j\|_{1,p}$$

which tends to 0 as $j \to \infty$. Conversely, if $C_0^\infty(G-K)$ is dense in $C_0^\infty(G)$ with respect to the norm $\|\cdot\|_{1,p}$, then any function $u \in C_0^\infty(G)$ such that $u=1$ on a neighborhood of K can be approximated by functions in $C_0^\infty(G-K)$, so that it follows that $C_p^*(K)=0$.

THEOREM 8.3. *Let G be a bounded open set in $\mathbf{R}^n$. If $E \subseteqq G$ and $C_p(E;G) < \infty$, then there exists a p-precise function u on G such that*

(i) $u \geqq 0$ *p-a.e. on* $\mathbf{R}^n$.

(ii) $u = 0$ *p-a.e. outside* G.

(iii) $u = 1$ *p-a.e. on* E.

(iv) $C_p(E;G) = \|\nabla u\|_p{}^p$.

(v) $\int_G (|\nabla u|^{p-2}\nabla u)\cdot(\nabla\varphi)dx = 0$ *for every $\varphi \in C_0^\infty(G)$ such that $\varphi = 0$ on a neighborhood of E.*

PROOF. Take a sequence $\{u_j\}$ of nonnegative p-precise functions on G such that $u_j \geqq 1$ p-a.e. on E, $u_j = 0$ p-a.e. outside G and

$$\lim_{j\to\infty} \|\nabla u_j\|_p{}^p = C_p(E;G).$$

Since G is bounded, we may assume that u_j and $D_i u_j$ converge weakly to u and f_i in $L^p(\mathbf{R}^n)$, respectively. Since $D_i u = f_i$ in the sense of distributions, it follows that $u \in W^{1,p}(\mathbf{R}^n)$; let u be p-precise on $\mathbf{R}^n$. Moreover,

$$C_p(E;G) \leqq \liminf_{j,k\to\infty} \|\nabla(u_j+u_k)/2\|_p{}^p \leqq C_p(E;G),$$

so that Clarkson's inequality shows that $\nabla u_j \to \nabla u$ in $L^p(\mathbf{R}^n)$ and

$$C_p(E;G) = \|\nabla u\|_p{}^p.$$

Clearly, u satisfies (i), (ii) and (iii). If $\varphi \in C_0^\infty(G)$ is nonnegative on a neighborhood of E and $t>0$, we have

$$C_p(E;G) \leqq \|\nabla(u+t\varphi)\|_p{}^p,$$

so that

$$0 \leqq \lim_{t\to 0} \frac{\|\nabla(u+t\varphi)\|_p{}^p - \|\nabla u\|_p{}^p}{t} = \int_G (|\nabla u|^{p-2}\nabla u)\cdot(\nabla\varphi)dx;$$

in particular, if $\varphi = 0$ on a neighborhood of E, then

$$\int_G |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dx = 0,$$

which shows (v).

We say that

$$\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$$

is the nonlinear Laplace operator of order p, or p-Laplacian.

As an application of Propositions 8.7 and 8.8, we discuss the removable singularities.

THEOREM 8.4. *Let K be a compact set in an open set G and $1 < p < \infty$. Then $C_{1,p}(K) = 0$ if and only if any solution $u \in BL_1(L^p(G-K))$, for which $\Delta_p u = 0$ on $G-K$ in the sense of distributions, can be extended to a solution on G.*

PROOF. First suppose $C_{1,p}(K) = 0$, and take a p-precise function u on $G-K$ for which $\Delta_p u = 0$ on $G-K$ in the sense of distributions. Consider any function $\tilde{u}$ which equals u on $G-K$. Since the projection of K to any coordinate plane has Hausdorff dimension at most $n-p$, $\tilde{u}$ is ACL on G and

$$\int_G (|\nabla\tilde{u}|^{p-2}\nabla\tilde{u})\cdot(\nabla\varphi)dx = \int_{G-K}(|\nabla u|^{p-2}\nabla u)\cdot(\nabla\varphi)dx = 0$$

for every $\varphi \in C_0^\infty(G)$ such that $\varphi = 0$ on a neighborhood of K. In view of Proposition 8.8, we see that $\Delta_p\tilde{u} = 0$ on G.

The converse follows from Theorem 8.3 readily.

THEOREM 8.5. *Let K be a compact set in an open set G and $1 < p < \infty$. Then $C_{1,p}(K) = 0$ if and only if any harmonic function u in $BL_1(L^{p'}(G-K))$ can be extended to a harmonic function on G.*

PROOF. First, suppose $C_{1,p}(K) > 0$ and G is a bounded open set which includes K. In view of Theorem 8.3, there exists $u \in \overline{C_0^\infty(G)}$ (in $W^{1,p}(\mathbf{R}^n)$) such that

$$\int (|\nabla u|^{p-2}\nabla u)\cdot(\nabla\varphi)dx = 0$$

for every $\varphi \in C_0^\infty(G)$ which vanishes on a neighborhood of K. Consider

$$U(x) = \int \nabla U_2(x-y)\cdot\mathbf{f}(y)dy,$$

where $\mathbf{f} = |\nabla u|^{p-2}\nabla u \in [L^{p'}(G)]^n$ and $\mathbf{f} = 0$ outside G. For any $\varphi \in C_0^\infty(G-K)$, we have

$$\begin{aligned}\int U\Delta\varphi dx &= \int\left(\int \nabla U_2(x-y)\Delta\varphi(x)dx\right)\cdot\mathbf{f}(y)dy\\ &= \int (c\nabla\varphi(y))\cdot\mathbf{f}(y)dy = 0,\end{aligned}$$

which implies that U is harmonic in $G-K$. If U is harmonic in G in the sense of distributions, then the above considerations imply that

$$\int U\Delta\varphi dx = c\int \nabla\varphi(y)\cdot\mathbf{f}(y)dy = 0,$$

for any $\varphi \in C_0^\infty(G)$. Thus we see that $\mathbf{f} = 0$, so that $C_p(K) = 0$, from which a contradiction follows.

Conversely, suppose $C_{1,p}(K) = 0$, and let $u \in BL_1(L^{p'}(G - K))$ be harmonic in $G - K$. Consider any function $\tilde{u}$ which equals u on $G - K$. Then, as in the proof of Theorem 8.4, we see that $\tilde{u} \in BL_1(L^{p'}(G))$. Take a sequence $\{\psi_j\}$ as in Proposition 8.7. Then, for $\varphi \in C_0^\infty(G)$,

$$\begin{aligned}\int \tilde{u}\Delta\varphi dx &= -\lim_{j\to\infty}\int \nabla\tilde{u}\cdot\nabla((1-\psi_j)\varphi)dx \\ &= -\lim_{j\to\infty}\int_{G-K} \nabla u\cdot\nabla((1-\psi_j)\varphi)dx \\ &= \lim_{j\to\infty}\int_{G-K} u\Delta((1-\psi_j)\varphi)dx = 0.\end{aligned}$$

Thus $\tilde{u}$ equals a harmonic function h on G almost everywhere, so that h is the required harmonic extension of u.

In general, we can show that $C_{m,p}(K) = 0$ if and only if there exists a sequence $\{\varphi_j\}$ in $C_0^\infty(G)$ such that $\varphi_j = 1$ on a neighborhood of K and $\|\nabla^m\varphi_j\|_p$ tends to zero. Using this fact, we can treat several types of removable singularities for polyharmonic functions.

An application of Theorem 6.3 in Chapter 5 shows the following.

THEOREM 8.6. *Let $1 < p < n$. If u is a BLD function in $BL_1(L^p(\mathbf{R}^n))$, then there exists a number c such that*

$$\lim_{x_1\to\infty} u(x_1, x') = c$$

for every $x' \in \mathbf{R}^{n-1} - E'$ with $C_{1,p}(\{0\}\times E') = 0$.

Finally we investigate Sobolev's inequalities for Beppo Levi functions.

THEOREM 8.7 (Gagliardo-Nirenberg-Sobolev inequality). *If $1 \leqq p < n$, then for any $u \in BL_1(L^p(\mathbf{R}^n))$, there exists a constant A for which*

$$\|u - A\|_q \leqq M\|\nabla u\|_p$$

when $1/q = 1/p - 1/n$.

PROOF. The case $1 < p < n$ follows from Sobolev's inequality, so that we treat the case $p = 1$. Then u is represented as

$$u(x) = c\sum_{j=1}^n \int (x_j - y_j)|x-y|^{-n} D_j u(y)dy + A.$$

Since

$$u(x) - A = \int_{-\infty}^{x_j} D_j u(x_1, ..., y_j, ..., x_n)dy_j,$$

we have

$$|u(x) - A| \leqq \int_{-\infty}^{\infty} |D_j u(x_1, ..., y_j, ..., x_n)| dy_j;$$

here we may assume that u is ACL, so that the inequality holds for almost every $(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)$. Hence, since $q = n/(n-1)$,

$$|u(x) - A|^q \leqq \prod_{j=1}^{n} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, ..., y_j, ..., x_n)| dy_j \right)^{1/(n-1)}.$$

Integrating both sides with respect to x_1, we have

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x) - A|^q dx_1 &\leqq \left(\int_{-\infty}^{\infty} |\nabla u| dy_1 \right)^{1/(n-1)} \int_{-\infty}^{\infty} \prod_{j=2}^{n} \left(\int_{-\infty}^{\infty} |\nabla u| dy_j \right)^{1/(n-1)} dx_1 \\ &\leqq \left(\int_{-\infty}^{\infty} |\nabla u| dy_1 \right)^{1/(n-1)} \prod_{j=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u| dy_j dx_1 \right)^{1/(n-1)}. \end{aligned}$$

Next, integrating both sides with respect to x_2, we find

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x) - A|^q dx_1 dx_2 &\leqq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u| dx_1 dy_2 \right)^{1/(n-1)} \\ &\quad \times \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u| dy_1 dx_2 \right)^{1/(n-1)} \\ &\quad \times \prod_{j=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u| dy_j dx_1 dx_2 \right)^{1/(n-1)}. \end{aligned}$$

Repeating these processes, we finally obtain the required inequality for $p = 1$.

THEOREM 8.8 (Poincaré's inequality on balls). *If $u \in W^{1,p}(B)$ and $1 \leqq p < n$, then*

$$\left(\frac{1}{|B|} \int_B |u(x) - u_B|^q dx \right)^{1/q} \leqq Mr \left(\frac{1}{|B|} \int_B |\nabla u(y)|^p dy \right)^{1/p}$$

when $1/q = 1/p - 1/n$, where $B = B(0, r)$ and

$$u_B = \frac{1}{|B|} \int_B u(y) dy.$$

PROOF. As in the proof of Theorem 7.5, we have

$$\left(\frac{1}{|B|} \int_B |u(x) - u_B|^p dx \right)^{1/p} \leqq Mr \left(\frac{1}{|B|} \int_B |\nabla u(y)|^p dy \right)^{1/p}. \tag{8.2}$$

For the case of Sobolev's exponent, we may assume that $r = 1$ and extend any $v \in W^{1,p}(\mathbf{B})$ to a function $\overline{v} \in W^{1,p}(\mathbf{R}^n)$ so that

$$\|\overline{v}\|_{1,p} \leqq M \|v\|_{W^{1,p}(\mathbf{B})};$$

for example, consider

$$\overline{v}(x) = \varphi(x) \times \begin{cases} v(x) & \text{when } |x| < 1, \\ v(x/|x|^2) & \text{when } |x| > 1 \end{cases}$$

for $\varphi \in C_0^\infty(B(0,3))$ such that $\varphi = 1$ on $B(0,2)$. Now, applying Sobolev's inequality, we find

$$\|\overline{v}\|_q \leqq M\|\nabla \overline{v}\|_p \leqq M\|v\|_{W^{1,p}(\mathbf{B})};$$

then consider $v = u - u_{\mathbf{B}}$ and apply (8.2).

Finally we show the following characterization for the Sobolev space $W^{1,p}(\mathbf{R}^n)$.

THEOREM 8.9. *A function $u \in L^p(\mathbf{R}^n)$ belongs to $W^{1,p}(\mathbf{R}^n)$ if and only if*

$$\|u(\cdot + h) - u(\cdot)\|_p \leqq M|h| \qquad \text{for every } h \in \mathbf{R}^n. \tag{8.3}$$

PROOF. If (8.3) holds, then Fatou's theorem implies that

$$\|\nabla u\|_p \leqq M.$$

Conversely, suppose $u \in W^{1,p}(\mathbf{R}^n)$. By considering approximation, we may assume that $u \in C_0^1(\mathbf{R}^n)$. Then we have

$$|u(x+h) - u(x)| \leqq |h| \int_0^1 |\nabla u(x + th)| dt.$$

Hence, Minkowski's inequality for integral gives

$$\|u(\cdot + h) - u(\cdot)\|_p \leqq |h| \int_0^1 \|\nabla u(\cdot + th)\|_p dt = |h|\|\nabla u\|_p,$$

which yields (8.3).

Chapter 7

Bessel potentials

This chapter concerns with the relationships between Bessel potential spaces and Lipschitz spaces. The Bessel kernels are given in the integral form and their Fourier transforms will be computed. The Bessel kernels behave like α-kernels near the origin, but decrease faster at infinity so that they are integrable on the whole space.

7.1 Bessel kernel

We first introduce the Bessel kernel g_α whose Fourier transform is given by

$$\hat{g}_\alpha(\xi) = \mathcal{F}g_\alpha(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}.$$

We show below that the function

$$g_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta} e^{-\delta/4\pi} \delta^{(\alpha-n)/2} \frac{d\delta}{\delta} \tag{1.1}$$

has the required property.

THEOREM 1.1. (1) *For every $\alpha > 0$, $g_\alpha \in L^1(\mathbf{R}^n)$.*

(2) $\hat{g}_\alpha(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}$.

PROOF. Noting that

$$\int e^{-\pi|x|^2/\delta}\, dx = \delta^{n/2},$$

we have by Fubini's theorem,

$$\int g_\alpha(x)\, dx = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\delta/4\pi} \delta^{\alpha/2} \frac{d\delta}{\delta} = 1,$$

which proves (1). According to Remark 2.1 in Chapter 2, note that

$$\mathcal{F}\left(e^{-\pi|x|^2/\delta}\right)(\xi) = e^{-\pi\delta|\xi|^2}\delta^{n/2}.$$

Hence Fubini's theorem gives

$$\mathcal{F}\left(\int_0^\infty e^{-\pi|x|^2/\delta}e^{-\delta/4\pi}\delta^{(\alpha-n)/2}\frac{d\delta}{\delta}\right)(\xi)=\int_0^\infty e^{-\pi\delta|\xi|^2}e^{-\delta/4\pi}\delta^{\alpha/2}\frac{d\delta}{\delta}.$$

On the other hand, since

$$t^{-\alpha/2}=\frac{1}{\Gamma(\alpha/2)}\int_0^\infty e^{-t\delta}\delta^{\alpha/2}\frac{d\delta}{\delta}, \tag{1.2}$$

we have by letting $t=(1+4\pi^2|\xi|^2)/4\pi$,

$$(1+4\pi^2|\xi|^2)^{-\alpha/2}=\hat{g}_\alpha(\xi),$$

as required.

For $0<\alpha<n$, by (1.2) we find

$$\frac{|x|^{\alpha-n}}{\gamma(\alpha)}=\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_0^\infty e^{-\pi|x|^2/\delta}\delta^{(\alpha-n)/2}\frac{d\delta}{\delta},\quad \gamma(\alpha)=\frac{\pi^{n/2}2^\alpha\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}.$$

Noting that $e^{-\delta/4\pi}=1+o(e^{-\delta/4\pi})$ as $\delta\to 0$, we establish

$$g_\alpha(x)=\frac{|x|^{\alpha-n}}{\gamma(\alpha)}+o(|x|^{\alpha-n})\qquad \text{as } |x|\to 0. \tag{1.3}$$

On the other hand, for $0<\varepsilon<1$ and $|x|\geqq 1$,

$$\int_1^\infty e^{-\pi|x|^2/\delta}e^{-\delta/4\pi}\delta^{(\alpha-n)/2}\frac{d\delta}{\delta}\leqq e^{-\sqrt{\varepsilon}|x|/2}\int_1^\infty e^{-(1-\varepsilon)\delta/4\pi}\delta^{(\alpha-n)/2}\frac{d\delta}{\delta}$$

and

$$\int_0^1 e^{-\pi|x|^2/\delta}e^{-\delta/4\pi}\delta^{(\alpha-n)/2}\frac{d\delta}{\delta}\leqq e^{-|x|^2}\int_0^1 e^{-1/\delta}\delta^{(\alpha-n)/2}\frac{d\delta}{\delta}.$$

Thus we see that for $0<c<1/2$,

$$g_\alpha(x)=O(e^{-c|x|})\qquad \text{as } |x|\to\infty. \tag{1.4}$$

Theorem 1.2. *$\{g_\alpha\}$ has the semigroup property, that is, for $\alpha>0$ and $\beta>0$,*

$$g_\alpha * g_\beta = g_{\alpha+\beta}.$$

7.2 Bessel potentials

For a function f, we define the Bessel potential

$$g_\alpha f(x) = g_\alpha * f(x) = \int g_\alpha(x-y)f(y)\,dy.$$

In view of Theorem 1.1 and Young's inequality, we have the following.

THEOREM 2.1. *For $\alpha > 0$ and $1 \leqq p \leqq \infty$, $\|g_\alpha f\|_p \leqq \|f\|_p$.*

For $\alpha > 0$ and $p \geqq 1$, we write

$$L^p_\alpha(\mathbf{R}^n) = g_\alpha(L^p(\mathbf{R}^n)) = \{g_\alpha f : f \in L^p(\mathbf{R}^n)\}$$

and

$$\|u\|_{L^p_\alpha} = \|f\|_p, \qquad u = g_\alpha f.$$

In view of Theorems 1.2 and 2.1, if $\alpha > \beta > 0$, then

$$L^p_\alpha(\mathbf{R}^n) \subseteqq L^p_\beta(\mathbf{R}^n) \subseteqq L^p(\mathbf{R}^n). \tag{2.1}$$

THEOREM 2.2. *If m is a positive integer and $1 < p < \infty$, then*

$$L^p_m(\mathbf{R}^n) = W^{m,p}(\mathbf{R}^n).$$

To prove this, we need several lemmas.

LEMMA 2.1. *For $\alpha > 0$, there exists a finite signed measure μ_α on $\mathbf{R}^n$ for which*

$$\hat{\mu}_\alpha(\xi) = \frac{(2\pi|\xi|)^\alpha}{(1+4\pi^2|\xi|^2)^{\alpha/2}}.$$

PROOF. For this purpose, note that

$$(1-t)^{\alpha/2} = 1 + \sum_{j=1}^\infty a_{j,\alpha} t^j \qquad \text{for } |t| < 1;$$

here $\sum_j |a_{j,\alpha}| < \infty$. If we take $t = 1/(1+4\pi^2|\xi|^2)$, then

$$\frac{(2\pi|\xi|)^\alpha}{(1+4\pi^2|\xi|^2)^{\alpha/2}} = 1 + \sum_{j=1}^\infty a_{j,\alpha}(1+4\pi^2|\xi|^2)^{-j}.$$

Now define

$$\mu = \delta_0 + \sum_{j=1}^\infty a_{j,\alpha} g_{2j},$$

where δ_0 denotes the Dirac measure at the origin. In view of Theorem 2.1, we see that

$$\mu(\mathbf{R}^n) = 1 + \sum_j a_{j,\alpha},$$

which is finite.

LEMMA 2.2. *For $\alpha > 0$, there exists a finite signed measure λ_α on $\mathbf{R}^n$ for which*

$$(1 + 4\pi^2|\xi|^2)^{\alpha/2} = \hat{\lambda}_\alpha[1 + (2\pi|\xi|)^\alpha].$$

PROOF. As above, note that

$$\Phi_1(x) = g_\alpha(x) + \sum_{j=1}^{\infty} a_{j,\alpha} g_{2j}(x) \in L^1(\mathbf{R}^n).$$

Further,

$$\hat{\Phi}_1(\xi) + 1 = \frac{(2\pi|\xi|)^\alpha + 1}{(1 + 4\pi^2|\xi|^2)^{\alpha/2}} > 0$$

for every x. Hence we apply the n-dimensional version of Wiener's theorem, which will be shown later, and obtain $\Phi_2 \in L^1(\mathbf{R}^n)$ such that

$$(1 + 4\pi^2|\xi|^2)^{\alpha/2} = [(2\pi|\xi|)^\alpha + 1][\hat{\Phi}_2(\xi) + 1].$$

Now, $\lambda_\alpha = \delta_0 + \Phi_2\mathcal{L}^n$ is the required one.

LEMMA 2.3. *If $u = g_\alpha * f$ with $\alpha \geqq 1$ and $f \in L^p(\mathbf{R}^n)$,*

$$\frac{\partial u}{\partial x_j} = g_{\alpha-1} * F_j, \qquad F_j = -R_j * (\mu_1 * f), \tag{2.2}$$

where R_j is the Riesz transform whose Fourier transform is just $-i\xi_j/|\xi|$.

In fact, if $f \in \mathcal{S}$, then (2.2) holds by taking the Fourier transforms of both sides. In the general case, approximate $f \in L^p(\mathbf{R}^n)$ by functions in $\mathcal{S}$.

For a multi-index $\boldsymbol{j} = (j_1, ..., j_n)$, define

$$\begin{aligned} \mathcal{R}^{\boldsymbol{j}} &= {R_1}^{j_1} * \cdots * {R_n}^{j_n} \\ &= (\overbrace{R_1 * \cdots * R_1}^{j_1}) * \cdots * (\overbrace{R_n * \cdots * R_n}^{j_n}), \end{aligned}$$

which is also characterized by

$$\mathcal{F}(\mathcal{R}^{\boldsymbol{j}} * f) = \left(-i\frac{y}{|y|}\right)^{\boldsymbol{j}} \hat{f}.$$

LEMMA 2.4. *If $u \in W^{m,p}(\mathbf{R}^n)$, then $u = g_m * f$ with $f \in L^p(\mathbf{R}^n)$ satisfying*

$$f = \lambda_m * \left(u + \sum_{|\boldsymbol{j}|=m} \frac{m!}{\boldsymbol{j}!} \mathcal{R}^{\boldsymbol{j}} * D^{\boldsymbol{j}} u \right).$$

PROOF OF THEOREM 2.2. First let $u = g_m f$ with $f \in L^p(\mathbf{R}^n)$. Then Lemma 2.3 implies that $\partial u/\partial x_j$ are all in $L^p_{m-1}(\mathbf{R}^n)$ and

$$\|\nabla u\|_p \leqq M\|f\|_p.$$

Hence it follows by induction that $u \in W^{m,p}(\mathbf{R}^n)$ and

$$\|u\|_{m,p} \leqq M\|u\|_{L^m_p}.$$

Conversely, assume that $u \in W^{m,p}(\mathbf{R}^n)$. Then we see from Lemma 2.4 that

$$u = g_m f$$

for some $f \in L^p(\mathbf{R}^n)$, so that

$$u \in L^p_m(\mathbf{R}^n).$$

Here note that

$$\|f\|_p \leqq M\|u\|_{m,p}.$$

What remains is to show Wiener's theorem mentioned above.

THEOREM 2.3 (Wiener's theorem). *If $\Phi_1 \in L^1(\mathbf{R}^n)$ and $\hat{\Phi}_1(x) + 1 \neq 0$ everywhere, then there exists $\Phi_2 \in L^1(\mathbf{R}^n)$ such that*

$$[\hat{\Phi}_1(x) + 1]^{-1} = \hat{\Phi}_2(x) + 1 \qquad \textit{for all } x.$$

PROOF. First, for any $\varepsilon > 0$ we find a function $g_0 \in L^1(\mathbf{R}^n)$ such that

$$\hat{g}_0 = 1 \qquad \text{on a neighborhood of } 0 \tag{2.3}$$

and

$$|[\hat{\Phi}_1(x) - \hat{\Phi}_1(0)]\hat{g}_0(x)| < \varepsilon \qquad \text{for all } x. \tag{2.4}$$

For this purpose, take $\psi \in \mathcal{S}(\mathbf{R}^n)$ for which $\hat{\psi} = 1$ on a neighborhood of 0. If we set

$$M_\rho f(x) = \rho^n f(\rho x)$$

for $f \in L^1(\mathbf{R}^n)$ and $\rho > 0$. Setting $a = \int f(x)dx$, we have

$$\begin{aligned} \|(M_\rho f) * \psi - a\psi\|_1 &= \left\| \int (M_\rho f)(z)[\psi(y-z) - \psi(y)]dz \right\|_1 \\ &= \left\| \int f(z)[\psi(y-\rho^{-1}z) - \psi(y)]dz \right\|_1 \\ &\leqq \int |f(z)| \left(\int |\psi(y-\rho^{-1}z) - \psi(y)|dy \right) dz, \end{aligned}$$

which tends to zero as $\rho \to \infty$, by Lebesgue's dominated convergence theorem. Since

$$\|(M_\rho f) * \psi - a\psi\|_1 = \|f * (M_{1/\rho}\psi) - aM_{1/\rho}\psi\|_1,$$

if ρ is large enough, then $g_0 = M_{1/\rho}\psi$ satisfies

$$|[\hat{f}(x) - \hat{f}(0)]\hat{g}_0(x)| \leqq \|f * g_0 - ag_0\|_1 < \varepsilon. \tag{2.5}$$

On the other hand, note that

$$\mathcal{F}(M_{1/\rho}\psi)(x) = \hat{\psi}(\rho x),$$

so that g_0 satisfies (2.3) and (2.4) by letting $f = \Phi_1$.

For any $x_0 \in \mathbf{R}^n$, if $\Phi_1(y)$ and $g_0(y)$ are replaced by $e^{-2\pi i x_0 \cdot y}\Phi_1(y) = f(y)$ and $e^{2\pi i x_0 \cdot y} g_0(y) = g_{x_0}(y)$, respectively, then

$$\hat{g}_{x_0}(x) = \hat{g}_0(x - x_0) = 1 \qquad \text{on a neighborhood of } x_0$$

and (2.5) implies that

$$|[\hat{\Phi}_1(x) - \hat{\Phi}_1(x_0)]\hat{g}_{x_0}(x)| = |[\hat{f}(x-x_0) - \hat{f}(0)]\hat{g}_0(x-x_0)| < \varepsilon.$$

Consider $A(z) = (z+1)^{-1} - 1$. Since $\hat{\Phi}_1 + 1 \neq 0$ and $\hat{\Phi}_1$ vanishes at infinity, A is holomorphic in a neighborhood of the closure of $\hat{\Phi}_1(\mathbf{R}^n)$. By the above considerations, there exists $g_0 \in \mathcal{S}$ such that $\hat{g}_0 \in C_0^\infty(\mathbf{R}^n)$, $\hat{g}_0 = 1$ on a neighborhood of 0 and

$$\|\Phi_1 - \Phi_1 * g_0\|_1 < \varepsilon.$$

Since $A(z) = (-z) + (-z)^2 + \cdots$ near the origin, we can find $h_0 \in L^1(\mathbf{R}^n)$ such that

$$\hat{h}_0(x) = \sum_{j=1}^{\infty} [-\hat{\Phi}_1(x)(1 - \hat{g}_0(x))]^j,$$

so that

$$\hat{h}_0 = A(\hat{\Phi}_1) \qquad \text{outside a compact set } K.$$

For $x_0 \in K$, write $z_0 = \hat{\Phi}_1(x_0)$ and

$$A(z) = A(z_0) + \sum_{j=1}^{\infty} (1+z_0)^{-j-1}(z_0 - z)^j.$$

If ε is chosen sufficiently small, then

$$\sum_{j=1}^{\infty} (1+z_0)^{-j-1}[-(\hat{\Phi}_1(x) - \hat{\Phi}_1(x_0))\hat{g}_{x_0}(x)]^j$$

converges in absolute value, so that there exists $h_{x_0} \in L^1(\mathbf{R}^n)$ for which

$$\hat{h}_{x_0}(x) = A(z_0)\hat{g}_{x_0}(x) + \sum_{j=1}^{\infty} c_n[-(\hat{\Phi}_1(x) - \hat{\Phi}_1(x_0))\hat{g}_{x_0}(x)]^j;$$

in particular,

$$\hat{h}_{x_0}(x) = A(\hat{\Phi}_1(x)) \qquad \text{on a neighborhood of } x_0.$$

Since K is compact, we can choose $x_j \in K$ and $h_j = h_{x_j} \in L^1(\mathbf{R}^n)$, $j = 1, 2, ..., N$, such that

$$\hat{h}_j(x) = A(\hat{\Phi}_1(x)) \qquad \text{on a neighborhood } U_j \text{ of } x_j$$

and

$$K \subseteqq \bigcup_{j=1}^{N} U_j.$$

For the convenience sake, let $U_0 = \mathbf{R}^n - K$, and take $\{e_j\}$ such that $\hat{e}_j \in C^\infty(U_j)$, $\hat{e}_j$ vanishes on a neighborhood of $\mathbf{R}^n - U_j$ and

$$\sum_{j=0}^{N} \hat{e}_j = 1 \qquad \text{on } \mathbf{R}^n.$$

Now consider

$$\Phi_2 = \sum_{j=0}^{N} e_j * h_j.$$

Then $\Phi_2 \in L^1(\mathbf{R}^n)$ and

$$\hat{\Phi}_2(x) = A(\hat{\Phi}_1(x)) \qquad \text{for all } x \in \mathbf{R}^n.$$

7.3 Bessel capacity

Let $1 < p < \infty$ and $\alpha > 0$. We define the Bessel capacity $B_{\alpha,p}$ by setting

$$B_{\alpha,p}(E) = \inf \ \|f\|_p{}^p,$$

where the infimum is taken over all nonnegative measurable functions f for which $g_\alpha f \geqq 1$ on E. Here α, p and (α, p) are called the order, the weight and the index of $B_{\alpha,p}$, respectively.

It is easy to see that $B_{\alpha,p}(E) = 0$ if and only if there exists a nonnegative function $f \in L^p(\mathbf{R}^n)$ such that

$$g_\alpha f(x) = g_\alpha * f(x) = \infty \qquad \text{for all } x \in E.$$

THEOREM 3.1. *The Bessel capacity $B_{\alpha,p}$ of order (α, p) is countably subadditive, nondecreasing and outer.*

THEOREM 3.2. *For $0 < \alpha < n$ and $E \subseteqq \mathbf{R}^n$, $C_{\alpha,p}(E) = 0$ if and only if $B_{\alpha,p}(E) = 0$.*

PROOF. First suppose $C_{\alpha,p}(E) = 0$. This means that $C_{\alpha,p}(E \cap D; D) = 0$ for any bounded open set D. If $C_{\alpha,p}(E \cap D; D) = 0$ for a bounded open set D, then we can find a nonnegative function $f \in L^p(\mathbf{R}^n)$ such that $f = 0$ outside D and

$$U_\alpha f(x) = \infty \qquad \text{for all } x \in E.$$

We see from (1.3) that

$$M^{-1} g_\alpha f(x) \leqq U_\alpha f(x) \leqq M g_\alpha f(x)$$

for any $x \in D$, so that it follows that $B_{\alpha,p}(E) = 0$. The converse is similarly proved with the aid of (1.4).

Theorem 1.4 in Chapter 5 gives the following result.

THEOREM 3.3. *If $\{E_j\}$ is a nondecreasing sequence of sets, then*

$$\lim_{j \to \infty} B_{\alpha,p}(E_j) = B_{\alpha,p}(\bigcup_j E_j).$$

We need another capacity $b_{\alpha,p}$, which is defined by setting

$$b_{\alpha,p}(E) = \sup \ \mu(E),$$

where the supremum is taken over all nonnegative measures μ for which the support of μ is contained in E and $\|g_\alpha \mu\|_p \leqq 1$.

Theorems 1.5 and 1.6 in Chapter 5 give the following results.

THEOREM 3.4. *If K is a compact set in $\mathbf{R}^n$, then*

$$B_{\alpha,p}(K) = [b_{\alpha,p}(K)]^p.$$

THEOREM 3.5. *Let A be a Suslin set in $\mathbf{R}^n$ for which $0 < B_{\alpha,p}(A) < \infty$. Then there exist $f \in L^p(\mathbf{R}^n)$ and $\mu \in \mathcal{M}(\overline{A})$ such that*

(i) $\|f\|_p^p = B_{\alpha,p}(A)$;

(ii) $g_\alpha f \geqq 1$ (α, p)-q.e. on A;

(iii) $\mu(\overline{A}) = b_{\alpha,p}(A)$;

(iv) $\|g_\alpha \mu\|_{L^{p'}(\mathbf{R}^n)} = 1$;

(v) $\mu(\{x : g_\alpha f(x) \neq 1\}) = 0$;

(vi) $g_\alpha \mu(y) = [B_{\alpha,p}(A)]^{-1/p'}[f(y)]^{p-1}$;

(vii) $g_\alpha([g_\alpha \mu]^{1/(p-1)}) \leqq [b_{\alpha,p}(A)]^{-1}$ on S_μ.

Weak maximum principle (see Theorem 4.1 in Chapter 5) implies that

$$g_\alpha([g_\alpha \mu]^{1/(p-1)}) \leqq M[b_{\alpha,p}(A)]^{-1} \qquad \text{on } \mathbf{R}^n.$$

7.4 Poisson kernel in the half space

Denote by $\mathbf{R}_+^{n+1}$ the upper half space of $\mathbf{R}^{n+1}$ whose point will be written for example as (x, t) with $x \in \mathbf{R}^n$ and $t > 0$, that is,

$$\mathbf{R}_+^{n+1} = \{(x, t) : x \in \mathbf{R}^n, t > 0\}.$$

We define the Poisson kernel for $\mathbf{R}_+^{n+1}$ by

$$P(x, t) = P_t(x) = \frac{c_n t}{(|x|^2 + t^2)^{(n+1)/2}}, \qquad c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}.$$

PROPOSITION 4.1. *For any $t > 0$,*

$$\int_{\mathbf{R}^n} P_t(x)dx = 1.$$

PROPOSITION 4.2. *For any $t > 0$, the Fourier transform of P_t is given by*

$$\hat{P}_t(y) = e^{-2\pi t|y|}.$$

PROOF. We need the following two identities :

$$\int_{\mathbf{R}^n} e^{-2\pi i x\cdot y} e^{-\pi\delta|y|^2} dy = \delta^{-n/2} e^{-\pi|x|^2/\delta}, \qquad \delta > 0$$

and

$$e^{-\gamma} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} e^{-\gamma^2/4s} ds, \qquad \gamma > 0.$$

To show the latter, we note that

$$\begin{aligned} e^{-\gamma} &= \frac{1}{\pi} \int_{-\infty}^\infty \frac{e^{i\gamma s}}{1+s^2} ds \\ &= \frac{1}{\pi} \int_{-\infty}^\infty e^{i\gamma s} \left(\int_0^\infty e^{-(1+s^2)u} du \right) ds \\ &= \frac{1}{\pi} \int_0^\infty e^{-u} \left(\int_{-\infty}^\infty e^{i\gamma s} e^{-s^2 u} ds \right) du, \end{aligned}$$

which gives the required identity. Now we have

$$\int_{\mathbf{R}^n} e^{-2\pi i x\cdot y} e^{-2\pi t|y|} dy = \frac{1}{\sqrt{\pi}} \int_{\mathbf{R}^n} \left(\int_0^\infty \frac{e^{-s}}{\sqrt{s}} e^{-\pi^2 t^2 |y|^2/s} ds \right) e^{-2\pi i x\cdot y} dy.$$

Here, applying the first identity with $\delta = \pi t^2/s$, we have

$$\begin{aligned} \int_{\mathbf{R}^n} e^{-2\pi i x\cdot y} e^{-2\pi t|y|} dy &= \frac{1}{\pi^{(n+1)/2}} \int_0^\infty e^{-s} e^{-|x|^2 s/t^2} t^{-n} s^{(n-1)/2} ds \\ &= \frac{t}{(\pi(|x|^2+t^2))^{(n+1/2)}} \int_0^\infty e^{-u} u^{(n-1)/2} du, \end{aligned}$$

which yields the required assertion.

Proposition 4.3. *$\{P_t\}$ has the semigroup property, that is,*

$$P_s * P_t = P_{s+t} \qquad \text{for any } s > 0 \text{ and } t > 0.$$

We define the Poisson integral of functions $f \in L^p(\mathbf{R}^n)$ by

$$P_t f(x) = P_t * f(x) = \int_{\mathbf{R}^n} P_t(x-y) f(y) dy.$$

Proposition 4.4. *If $f \in L^p(\mathbf{R}^n)$, $1 \leqq p \leqq \infty$, then*

$$\|P_t f\|_p \leqq \|f\|_p.$$

In case $1 \leqq p < \infty$, $P_t f \to f$ in $L^p(\mathbf{R}^n)$ as $t \to +0$. If f is continuous and vanishes at infinity, then $P_t f \to f$ uniformly as $t \to +0$.

As in the proof of Theorem 3.1 of Chapter 3, we can prove the following.

Theorem 4.1. *If $f \in L^p(\mathbf{R}^n)$, $1 \leqq p \leqq \infty$, then $u(x,t) = P_t f(x)$ is harmonic in $\mathbf{R}^{n+1}_+$ and*

$$\sup_{t>0} |P_t f(x)| \leqq M f(x).$$

Further,

$$\lim_{t \to 0} P_t f(x) = f(x) \qquad \textit{for almost every } x; \tag{4.1}$$

in particular, (4.1) *holds for x at which f is continuous.*

PROOF. By Proposition 4.1, we have the first assertion and

$$\begin{aligned}|u(x,t) - f(x)| &= \left| \int P(y,t)[f(x-y) - f(x)]dy \right| \\ &\leqq \int P(y,t)|f(x-y) - f(x)|dy \\ &\leqq \int_{B(0,r)} P(y,t)|f(x-y) - f(x)|dy \\ &\quad + \int_{\mathbf{R}^n - B(0,r)} P(y,t)|f(x-y) - f(x)|dy = I_1 + I_2.\end{aligned}$$

For $r > 0$, set

$$\varepsilon(r) = \sup_{0<s<r} \frac{1}{|B(0,s)|} \int_{B(0,s)} |f(x-y) - f(x)|dy.$$

If $0 < t < r$, then we have

$$\begin{aligned}I_1 &\leqq c_n r^{-n} \int_{B(0,r)} |f(x-y) - f(x)|dy \\ &\quad - \int_0^r \left(\int_{B(0,s)} |f(x-y) - f(x)|dy \right) d(c_n t/(s+t)^{n+1}) \leqq M\varepsilon(r).\end{aligned}$$

On the other hand,

$$\begin{aligned}I_2 &\leqq \int_{\mathbf{R}^n - B(0,r)} P(y,t)|f(x-y)|dy \\ &\quad + |f(x)| \int_{\mathbf{R}^n - B(0,r)} P(y,t)dy \\ &\leqq [Mr^{-1-n/p}]t\|f\|_p + |f(x)| \int_{\mathbf{R}^n - B(0,r)} P(y,t)dy,\end{aligned}$$

which tends to zero as $t \to +0$ for fixed $r > 0$. Thus

$$\lim_{t \to 0} |u(x,t) - f(x)| = 0$$

at every Lebesgue point x, that is,

$$\lim_{r \to 0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x-y) - f(x)|dy = 0; \tag{4.2}$$

in particular, (4.2) holds when f is continuous at x.

If $f \in \Lambda_\alpha(\mathbf{R}^n)$ and $0 < \alpha < 1$, then recall that

$$(4.3) \qquad |f(x+h) - f(x)| \leqq A|h|^\alpha \qquad \text{for all } h \in \mathbf{R}^n.$$

Hence we see that

$$\begin{aligned} |P_t f(x) - f(x)| &\leqq \left| \int P_t(y)[f(x-y) - f(x)]dy \right| \\ &\leqq c_n A t \int \frac{|y|^\alpha}{(|y|^2 + t^2)^{(n+1)/2}} dy = M t^\alpha, \end{aligned}$$

so that f can be extended to a continuous function on $\mathbf{R}^{n+1}$.

LEMMA 4.1. *Let $0 < \alpha < 1$ and $u(x,t) = P_t f$ for $f \in L^\infty(\mathbf{R}^n)$. Then*

$$(4.4) \qquad \left| \frac{\partial u(x,t)}{\partial t} \right| \leqq M t^{\alpha - 1}$$

holds if and only if

$$(4.5) \qquad |\nabla u(x,t)| \leqq M t^{\alpha-1},$$

where $\nabla = (\partial/\partial x_1, ..., \partial/\partial x_n)$.

PROOF. For $t = t_1 + t_2$, $t_1 > 0$, $t_2 > 0$, we see that

$$u(x,t) = P_{t_1} * u(x, t_2),$$

so that

$$\frac{\partial^2 u(x, t_1 + t_2)}{\partial x_j \partial t_2} = \frac{\partial P_{t_1}}{\partial x_j} * \frac{\partial u(x, t_2)}{\partial t_2}.$$

Note here that

$$(4.6) \qquad \left| \frac{\partial P_t(x)}{\partial x_j} \right| \leqq M(|x| + t)^{-n-1}.$$

Hence (4.4) implies that

$$\left| \frac{\partial^2 u(x, t_1 + t_2)}{\partial x_j \partial t_2} \right| \leqq M t_2^{\alpha-1} \int (|y| + t_1)^{-(n+1)} dy.$$

If we take $t_1 = t_2 = t/2$, then

$$(4.7) \qquad \left| \frac{\partial^2 u(x,t)}{\partial x_j \partial t} \right| \leqq M t^{\alpha-2}.$$

On the other hand,

$$|\nabla u(x,t)| \leqq |(\nabla P_t) * f| \leqq \|\nabla P_t\|_1 \|f\|_\infty \leqq M t^{-1} \|f\|_\infty,$$

so that

$$|\nabla u(x,t)| \to 0 \qquad \text{as } t \to \infty.$$

Thus we can write

$$\frac{\partial u(x,t)}{\partial x_j} = -\int_t^\infty \frac{\partial^2 u(x,s)}{\partial x_j \partial s}\, ds,$$

which gives (4.5) with the aid of (4.7).

Conversely, if (4.5) holds, then, by the above considerations, we find

$$|\nabla^2 u(x,t)| \leqq Mt^{\alpha-2}.$$

Since u is harmonic in $\mathbf{R}_+^{n+1}$, we have

$$\left|\frac{\partial^2 u(x,t)}{\partial t^2}\right| = \left|-\sum_{j=1}^n \frac{\partial^2 u(x,t)}{\partial x_j^2}\right| \leqq Mt^{\alpha-2}.$$

The above integration argument shows (4.4).

THEOREM 4.2. *Let $0 < \alpha < 1$ and $u(x,t) = P_t f(x)$ for $f \in L^\infty(\mathbf{R}^n)$. Then $f \in \Lambda_\alpha(\mathbf{R}^n)$ if and only if (4.4) holds.*

PROOF. Suppose (4.3) holds. Since $\displaystyle\int P_t(y)dy = 1$,

$$\int \frac{\partial P_t(y)}{\partial t}\, dy = 0, \tag{4.8}$$

so that

$$\frac{\partial u(x,t)}{\partial t} = \int \frac{\partial P_t(y)}{\partial t}[f(x-y) - f(x)]dy.$$

Since

$$\left|\frac{\partial P_t(x)}{\partial t}\right| \leqq M(|x|+t)^{-n-1}, \tag{4.9}$$

we find

$$\left\|\frac{\partial u(x,t)}{\partial t}\right\|_\infty \leqq M\int (|y|+t)^{-n-1}|y|^\alpha dy = Mt^{\alpha-1}.$$

Conversely, suppose (4.4) holds. Then (4.5) also holds by Lemma 4.1. We write

$$f(x+y) - f(x) = [u(x+y,t) - u(x,t)] + [f(x+y) - u(x+y,t)] - [f(x) - u(x,t)].$$

Then by taking $t = |y|$, we obtain

$$|u(x+y,t) - u(x,t)| \leqq |y| \sup_{0<\theta<1} |\nabla u(x+\theta y,t)| \leqq M|y|^\alpha.$$

Moreover,

$$|f(x+y)-u(x+y,t)| = \left|-\int_0^t \frac{\partial u(x+y,s)}{\partial s}\, ds\right| \leqq M\int_0^t s^{\alpha-1}ds \leqq Mt^{\alpha}.$$

Similarly,

$$|f(x)-u(x,t)| \leqq Mt^{\alpha}.$$

Thus (4.3) holds.

The proof of Lemma 4.1 shows the following result.

LEMMA 4.2. *Let $\alpha > 0$ and $u(x,t) = P_t * f(x)$ for $f \in L^\infty(\mathbf{R}^n)$. If k and ℓ are integers greater than α, then*

$$\left|\frac{\partial^k u(x,t)}{\partial t^k}\right| \leqq Mt^{\alpha-k} \qquad \text{and} \qquad \left|\frac{\partial^\ell u(x,t)}{\partial t^\ell}\right| \leqq Mt^{\alpha-\ell}$$

are equivalent.

THEOREM 4.3. *Let $0 < \alpha < 2$ and $f \in L^\infty(\mathbf{R}^n)$. Then*

$$|f(x+h)-2f(x)+f(x-h)| \leqq A|h|^{\alpha} \tag{4.10}$$

if and only if

$$\left|\frac{\partial^2 u(x,t)}{\partial t^2}\right| \leqq Mt^{\alpha-2}. \tag{4.11}$$

PROOF. Noting that

$$\int \frac{\partial^2 P_t(y)}{\partial t^2}\, dy = 0,$$

we have

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{1}{2}\int \frac{\partial^2 P_t(y)}{\partial t^2}[f(x+y)+f(x-y)-2f(x)]dy.$$

Hence (4.10) implies that

$$\left|\frac{\partial^2 u(x,t)}{\partial t^2}\right| \leqq M\int (|y|+t)^{-n-2}|y|^{\alpha}dy \leqq Mt^{\alpha-2},$$

which shows (4.11).

Next suppose (4.11) holds. Note that

$$\left|\frac{\partial^2 u(x,t)}{\partial t^2}\right| \leqq \left\|\frac{\partial^2 P_t}{\partial t^2}\right\|_1 \|f\|_\infty \leqq Mt^{-2}\|f\|_\infty,$$

so that

$$\left|\frac{\partial^2 u(x,t)}{\partial t^2}\right| \leqq M t^{\beta-2}$$

for $0 < \beta < \min\{\alpha, 1\}$. Hence Lemma 4.2 yields

$$\left|\frac{\partial u(x,t)}{\partial t}\right| \leqq M t^{\beta-1}.$$

Hence Theorem 4.2 shows that $f \in \Lambda_\beta$, so that

$$\|u(x,t) - f(x)\|_\infty \to 0 \qquad \text{as } t \to 0$$

and

$$t\left\|\frac{\partial u(x,t)}{\partial t}\right\|_\infty \to 0 \qquad \text{as } t \to 0.$$

Now we see that

$$f(x) = u(x,0) = \int_0^t s\frac{\partial^2 u(x,s)}{\partial s^2}\, ds - t\frac{\partial u(x,t)}{\partial t} + u(x,t). \tag{4.12}$$

As in the proof of Lemma 4.1, (4.11) shows

$$|\nabla^2 u(x,t)| \leqq M t^{\alpha-2}$$

and

$$|(\partial/\partial t)\nabla^2 u(x,t)| \leqq M t^{\alpha-3}.$$

Hence it follows from (4.12) that

$$|f(x+y) + f(x-y) - 2f(x)| \leqq M\int_0^t s^{\alpha-1}\, ds + Mt|y|^2 t^{\alpha-3} + M|y|^2 t^{\alpha-2}.$$

If we take $t = |y|$, then (4.10) follows.

Corollary 4.1. *Let $0 < \alpha < 1$ and $f \in L^\infty(\mathbf{R}^n)$. Then (4.3) holds if and only if*

$$|f(x+h) - 2f(x) + f(x-h)| \leqq A|h|^\alpha \qquad \textit{for all } h \in \mathbf{R}^n.$$

Remark 4.1. Consider the function

$$f(x) = x\log|x| \qquad \text{for } x \neq 0;$$

set $f(0) = 0$. Then f is continuous on $\mathbf{R}$. Clearly, f fails to satisfy (4.3) with $\alpha = 1$, but it satisfies

$$|f(x+y) - 2f(x) + f(x-y)| \leqq A|y|.$$

In fact, if $x > 0$ and $y > 0$, then we write

$$\begin{aligned} f(x+y) - 2f(x) + f(x-y) &= -(x-y)[\log|x+y| - \log|x-y|] \\ &\quad +2x[\log|x+y| - \log|x|] \end{aligned}$$

and note that

$$\log(1+t) \leqq Mt \qquad \text{for } t > 0.$$

For $\alpha > 0$, take the nonnegative integer k such that

$$k < \alpha \leqq k+1.$$

Let $f \in L^\infty(\mathbf{R}^n) \cap C^k(\mathbf{R}^n)$. Then recall that $f \in \Lambda_\alpha(\mathbf{R}^n)$ if

$$|\nabla^k f(x+h) - 2\nabla^k f(x) + \nabla^k f(x-h)| \leqq A|h|^{\alpha-k} \text{ for all } h \in \mathbf{R}^n. \tag{4.13}$$

Theorem 4.4 can be generalized as follows.

THEOREM 4.5. *For $\alpha > 0$, $f \in \Lambda_\alpha(\mathbf{R}^n)$ if and only if*

$$\left|\frac{\partial^\ell u(x,t)}{\partial t^\ell}\right| \leqq Mt^{\alpha-\ell} \tag{4.14}$$

for some integer $\ell > \alpha$.

7.5 The Lipschitz spaces $\Lambda_\alpha^{p,q}$

For $0 < \alpha < 1$, $1 \leqq p \leqq \infty$ and $1 \leqq q \leqq \infty$, we denote by $\Lambda_\alpha^{p,q} = \Lambda_\alpha^{p,q}(\mathbf{R}^n)$ the space of all functions $f \in L^p(\mathbf{R}^n)$ such that

$$\|f\|_{\Lambda_\alpha^{p,q}} = \|f\|_p + \left(\int_{\mathbf{R}^n} \frac{\|f(x+y) - f(x)\|_p{}^q}{|y|^{n+\alpha q}}\, dy\right)^{1/q} < \infty.$$

THEOREM 5.1. *Let $f \in L^p(\mathbf{R}^n)$ and $0 < \alpha < 1$. Then $f \in \Lambda_\alpha^{p,q}$ if and only if*

$$\left(\int_0^\infty [t^{1-\alpha}\|(\partial/\partial t)u(x,t)\|_p]^q\, dt/t\right)^{1/q} < \infty, \tag{5.1}$$

where $u(x,t) = P_t f(x)$ is the Poisson integral of f in $\mathbf{R}_+^{n+1}$. Moreover

$$\|f\|_{\Lambda_\alpha^{p,q}} \sim \|f\|_p + \left(\int_0^\infty [t^{1-\alpha}\|(\partial/\partial t)u(x,t)\|_p]^q\, dt/t\right)^{1/q}.$$

PROOF. By (4.8), we see that

$$(\partial/\partial t)u(x,t) = \int [(\partial/\partial t)P_t(y)]f(x+y)\ dy = \int [(\partial/\partial t)P_t(y)][f(x+y)-f(x)]dy.$$

Note by (4.9) that

$$|(\partial/\partial t)P_t(y)| \leqq Mt^{-n-1}$$

and

$$|(\partial/\partial t)P_t(y)| \leqq M|y|^{-n-1}.$$

Hence we have

$$\begin{aligned}\|(\partial/\partial t)u(x,t)\|_p &\leqq Mt^{-n-1}\int_{B(0,t)} \|f(x+y)-f(x)\|_p\ dy\\ &\quad +M\int_{\mathbf{R}^n-B(0,t)} \|f(x+y)-f(x)\|_p|y|^{-n-1}dy.\end{aligned}$$

We write $y = r\xi$ with $r = |y|$ and $|\xi| = 1$. Set

$$\omega_p(y) = \|f(x+y)-f(x)\|_p$$

and

$$\Omega_p(r) = \int_{\mathbf{S}} \omega_p(r\xi)\ dS(\xi).$$

Then

$$\|(\partial/\partial t)u(x,t)\|_p \leqq Mt^{-n-1}\int_0^t \Omega_p(r)r^{n-1}dr + M\int_t^\infty \Omega_p(r)r^{-2}dr.$$

We now apply Hardy's inequality to obtain

$$\left(\int_0^\infty [t^{1-\alpha}\|(\partial/\partial t)u(x,t)\|_p]^q\ dt/t\right)^{1/q} \leqq M\left(\int_0^\infty [\Omega_p(r)r^{-\alpha}]^q\ dr/r\right)^{1/q}.$$

Since $[\Omega_p(r)]^q \leqq M\int_{\mathbf{S}} \omega_p(r\xi)^q\ dS(\xi)$ by Hölder's inequality, we establish

$$\begin{aligned}\left(\int_0^\infty [\Omega_p(r)r^{-\alpha}]^q\ dr/r\right)^{1/q} &\leqq M\left(\int_{\mathbf{S}}\int_0^\infty [\omega_p(r\xi)r^{-\alpha}]^q\ dS(\xi)dr/r\right)^{1/q}\\ &= M\left(\int_{\mathbf{R}^n} \frac{\|f(x+y)-f(x)\|_p{}^q}{|y|^{n+\alpha q}}\ dy\right)^{1/q}.\end{aligned}$$

To show the converse, we need the following result.

LEMMA 5.1. *Let $f \in L^p(\mathbf{R}^n)$ and $0 < \alpha < 1$. Then (5.1) holds if and only if*

$$\left(\int_0^\infty [t^{1-\alpha}\|(\partial/\partial x_i)u(x,t)\|_p]^q\ dt/t\right)^{1/q} < \infty, \qquad i = 1,2,...,n.$$

PROOF. For $t = t_1 + t_2$, $t_1 > 0$, $t_2 > 0$, we find

$$\frac{\partial^2 u}{\partial t \partial x_i}(x,t) = \frac{\partial P_{t_1}}{\partial x_i} * \frac{\partial u(x,t_2)}{\partial t_2}.$$

If we take $t_1 = t_2 = t/2$, then

$$\left\| \frac{\partial^2 u}{\partial t \partial x_i} \right\|_p \leqq \frac{M}{t} \left\| \frac{\partial u}{\partial t} \right\|_p, \tag{5.2}$$

so that

$$\left(\int_0^\infty [t^{2-\alpha} \|(\partial^2/\partial t \partial x_i)u\|_p]^q \, dt/t \right)^{1/q} \leqq M \left(\int_0^\infty [t^{1-\alpha} \|(\partial/\partial t)u\|_p]^q \, dt/t \right)^{1/q}.$$

On the other hand,

$$\begin{aligned} |(\partial/\partial x_i)u(x,t)| &= \left| \int [(\partial/\partial x_i)P_t(x-y)]f(y)dy \right| \\ &\leqq \|(\partial/\partial x_i)P_t\|_{p'} \|f\|_p \\ &\leqq Mt^{-1-n/p}\|f\|_p \to 0 \qquad \text{as } t \to \infty. \end{aligned}$$

Hence we find

$$(\partial/\partial x_i)u(x,t) = -\int_t^\infty (\partial^2/\partial s \partial x_i)u(x,s) \, ds.$$

It follows from Minkowski's inequality for integral that

$$\|(\partial/\partial x_i)u(x,t)\|_p \leqq \int_t^\infty \|(\partial^2/\partial s \partial x_i)u(x,s)\|_p \, ds.$$

Therefore we have by Hardy's inequality,

$$\begin{aligned} \left\{ \int_0^\infty \left(t^{1-\alpha} \left\| \frac{\partial u}{\partial x_i} \right\|_p \right)^q \frac{dt}{t} \right\}^{1/q} &\leqq M \left\{ \int_0^\infty \left(t^{1-\alpha} \int_t^\infty \left\| \frac{\partial^2 u}{\partial s \partial x_i} \right\|_p ds \right)^q \frac{dt}{t} \right\}^{1/q} \\ &\leqq M \left(\int_0^\infty [t \|\partial^2 u/\partial t \partial x_i\|_p]^q t^{(1-\alpha)q} \, dt/t \right)^{1/q} \\ &= M \left(\int_0^\infty [t^{2-\alpha} \|\partial^2 u/\partial t \partial x_i\|_p]^q \, dt/t \right)^{1/q} \\ &\leqq M \left(\int_0^\infty [t^{1-\alpha} \|\partial u/\partial t\|_p]^q \, dt/t \right)^{1/q}. \end{aligned}$$

As in (5.2), note next that

$$\|(\partial^2/\partial x_i^2)u\|_p \leqq \frac{M}{t} \|(\partial/\partial x_i)u\|_p,$$

which gives

$$\left(\int_0^\infty [t^{2-\alpha}\|(\partial^2/\partial x_i^2)u\|_p]^q\ dt/t\right)^{1/q} \leqq M\left(\int_0^\infty [t^{1-\alpha}\|(\partial/\partial x_i)u\|_p]^q\ dt/t\right)^{1/q}.$$

Since u is harmonic,

$$\|(\partial^2/\partial t^2)u\|_p = \left\|\sum_{i=1}^n (\partial^2/\partial x_i^2)u\right\|_p \leqq \sum_i \|(\partial^2/\partial x_i^2)u\|_p,$$

so that

$$\left(\int_0^\infty [t^{2-\alpha}\|(\partial^2/\partial t^2)u\|_p]^q\ dt/t\right)^{1/q} \leqq M\sum_i\left(\int_0^\infty [t^{1-\alpha}\|(\partial/\partial x_i)u\|_p]^q\ dt/t\right)^{1/q}.$$

As above we finally obtain

$$\begin{aligned}\left(\int_0^\infty [t^{1-\alpha}\|(\partial/\partial t)u\|_p]^q\ dt/t\right)^{1/q} &\leqq M\left(\int_0^\infty [t^{2-\alpha}\|(\partial^2/\partial t^2)u\|_p]^q\ dt/t\right)^{1/q}\\ &\leqq M\sum_i\left(\int_0^\infty [t^{1-\alpha}\|(\partial/\partial x_i)u\|_p]^q\ dt/t\right)^{1/q}.\end{aligned}$$

We are now ready to show the if part of Theorem 5.1. For this purpose, note that

$$f(x) = u(x,0) = -\int_0^t (\partial/\partial s)u(x,s)ds + u(x,t),$$

so that

$$f(x+y) - f(x) = -\int_0^t [(\partial/\partial s)u(x+y,s) - (\partial/\partial s)u(x,s)]ds + [u(x+y,t) - u(x,t)].$$

Since $|u(x+y,t) - u(x,t)| \leqq |y|\int_0^1 |\nabla u(x+sy,t)|ds$, we have

$$\|u(x+y,t) - u(x,t)\|_p \leqq |y|\ \|\nabla u(x,t)\|_p.$$

Letting $t = |y|$, we obtain

$$\begin{aligned}\left(\int \frac{\|u(x+y,t) - u(x,t)\|_p{}^q}{|y|^{n+\alpha q}}\ dy\right)^{1/q} &\leqq M\left(\int_0^\infty \frac{[t\|\nabla u(x,t)\|_p]^q}{t^{n+\alpha q}}\ t^{n-1}dt\right)^{1/q}\\ &= M\left(\int_0^\infty [t^{1-\alpha}\|\nabla u(x,t)\|_p]^q\ dt/t\right)^{1/q}\end{aligned}$$

and, by Hardy's inequality,

$$\begin{aligned}&\left(\int \left\|-\int_0^t [(\partial/\partial s)u(x+y,s) - (\partial/\partial s)u(x,s)]ds\right\|_p^q |y|^{-n-\alpha q}dy\right)^{1/q}\\ \leqq\ & 2\left\{\int\left(\int_0^t \|(\partial/\partial s)u(x,s)\|_p ds\right)^q |y|^{-n-\alpha q}dy\right\}^{1/q}\\ =\ & M\left\{\int_0^\infty\left(\int_0^t \|(\partial/\partial s)u(x,s)\|_p ds\right)^q t^{-\alpha q}dt/t\right\}^{1/q}\\ \leqq\ & M\left\{\int_0^\infty [t^{1-\alpha}\|(\partial/\partial t)u(x,t)\|_p]^q\ dt/t\right\}^{1/q}.\end{aligned}$$

Remark 5.1. Let $f \in L^p(\mathbf{R}^n)$ and $0 < \alpha < 1$. If k is a positive integer such that $k > \alpha$, then

$$\begin{aligned} &\left(\int_0^\infty [t^{k-\alpha}\|\nabla^k u(x,t)\|_p]^q \; dt/t\right)^{1/q} \\ \leqq \; & M\left(\int_0^\infty [t^{k-\alpha}\|(\partial/\partial t)^k u(x,t)\|_p]^q \; dt/t\right)^{1/q} \\ \leqq \; & M\left(\int_0^\infty [t^{1-\alpha}\|(\partial/\partial t) u(x,t)\|_p]^q \; dt/t\right)^{1/q}. \end{aligned}$$

For $\alpha > 0$, taking the smallest integer ℓ such that $\ell > \alpha$, we define

$$\Lambda_\alpha^{p,q} = \left\{ f \in L^p(\mathbf{R}^n) : \int_0^\infty [t^{\ell-\alpha}\|(\partial/\partial t)^\ell u(x,t)\|_p]^q \; dt/t < \infty \right\}$$

and

$$\|f\|_{\Lambda_\alpha^{p,q}} = \|f\|_p + \left(\int_0^\infty [t^{k-\alpha}\|(\partial/\partial t)^k u(x,t)\|_p]^q dt/t\right)^{1/q}.$$

Lemma 5.2. *If* $0 < \alpha < \beta < \infty$, *then* $\Lambda_\beta^{p,q} \subseteqq \Lambda_\alpha^{p,q}$.

Proof. If $\beta > \alpha$ and k is a positive integer greater than β, then

$$\int_0^1 [t^{k-\alpha}\|(\partial/\partial t)^k u(x,t)\|_p]^q dt/t \leqq \int_0^1 [t^{k-\beta}\|(\partial/\partial t)^k u(x,t)\|_p]^q dt/t < \infty,$$

where $u = P_t f$ with $f \in \Lambda_\beta^{p,q}$. On the other hand, we see that

$$\|(\partial/\partial t)^k P_t f\|_p \leqq \|(\partial/\partial t)^k P_t\|_1 \|f\|_p \leqq M t^{-k}\|f\|_p,$$

so that

$$\int_1^\infty [t^{k-\alpha}\|(\partial/\partial t)^k u(x,t)\|_p]^q dt/t < \infty.$$

Hence it follows that

$$\int_0^\infty [t^{k-\alpha}\|(\partial/\partial t)^k u(x,t)\|_p]^q dt/t < \infty,$$

which implies that $f \in \Lambda_\alpha^{p,q}$ in view of Remark 5.1.

Theorem 5.2. *Let* $f \in L^p(\mathbf{R}^n)$ *and* $0 < \alpha < 2$. *Then* $f \in \Lambda_\alpha^{p,q}$ *if and only if*

$$\int_{\mathbf{R}^n} \frac{\|f(x+y) - 2f(x) + f(x-y)\|_p{}^q}{|y|^{n+\alpha q}} \; dy < \infty$$

and

$$\|f\|_{\Lambda_\alpha^{p,q}} \sim \|f\|_p + \left(\int_{\mathbf{R}^n} \frac{\|f(x+y) - 2f(x) + f(x-y)\|_p{}^q}{|y|^{n+\alpha q}} \; dy\right)^{1/q}.$$

PROOF. The proof can be carried out along the same lines as in the proof of Theorem 5.1. Since $\int (\partial^2/\partial t^2)P_t(y)\,dy = 0$, we find

$$(\partial^2/\partial t^2)u(x,t) = \frac{1}{2}\int [(\partial^2/\partial t^2)P_t(y)][f(x+y)-2f(x)+f(x-y)]dy.$$

Hence we have

$$\|(\partial^2/\partial t^2)u(x,t)\|_p \leqq Mt^{-n-2}\int_{B(0,t)} \tilde{\omega}_p(y)dy + M\int_{\mathbf{R}^n - B(0,t)} \frac{\tilde{\omega}_p(y)}{|y|^{n+2}}\,dy,$$

where $\tilde{\omega}_p(y) = \|f(x+y)-2f(x)+f(x-y)\|_p$. Setting

$$\tilde{\Omega}_p(r) = \int_{\mathbf{S}} \tilde{\omega}_p(r\xi)\,dS(\xi),$$

we obtain

$$\|(\partial^2/\partial t^2)u(x,t)\|_p \leqq Mt^{-n-2}\int_0^t \tilde{\Omega}_p(r)r^{n-1}dr + M\int_t^\infty \tilde{\Omega}_p(r)r^{-3}dr.$$

We now apply Hardy's inequality to obtain

$$\left(\int_0^\infty [t^{2-\alpha}\|(\partial^2/\partial t^2)u\|_p]^q\,dt/t\right)^{1/q} \leqq M\left(\int_0^\infty [\tilde{\Omega}_p(r)r^{-\alpha}]^q\,dr/r\right)^{1/q}$$
$$\leqq M\left(\int_{\mathbf{R}^n} \frac{\|f(x+y)-2f(x)+f(x-y)\|_p{}^q}{|y|^{n+\alpha q}}\,dy\right)^{1/q}.$$

For $y \in \mathbf{R}^n$ and $F \in C^2$, set

$$\Delta_y^2 F(x) = F(x+y) - 2F(x) + F(x-y).$$

Then note that

$$\Delta_h^2 F(x) = \int_0^{|y|}\left(\int_{-s}^s (\partial^2/\partial r^2)F(x+ry')dr\right)ds, \qquad y' = y/|y|,$$

so that

$$\|\Delta_y^2 F\|_p \leqq |y|^2\|\nabla^2 F\|_p.$$

Since $t|(\partial/\partial t)u(x,t)| \to 0$ as $t \to +0$, we see that

$$f(x) = u(x,0) = -\int_0^t s[(\partial^2/\partial s^2)u(x,s)]ds - t[(\partial/\partial t)u(x,t)] + u(x,t).$$

Consequently, we have

$$\begin{aligned}\|\Delta_y^2 f\|_p &\leqq \int_0^t s\|\Delta_y^2((\partial^2/\partial s^2)u(x,s))\|_p ds + t\|\Delta_y^2((\partial/\partial t)u(x,t))\|_p \\ &\quad +\|\Delta_y^2(u(x,t))\|_p \\ &\leqq 4\int_0^t s\|(\partial^2/\partial s^2)u(x,s)\|_p ds + Mt\sum_{i,j}\|(\partial^3/\partial t\partial x_i\partial x_j)u(x,t))\|_p \\ &\quad +M\sum_{i,j}\|(\partial^2/\partial x_i\partial x_j)u(x,t))\|_p.\end{aligned}$$

Letting $t = |y|$, we establish

$$\begin{aligned}
\left(\int \|\Delta_y^2 f\|_p{}^q |y|^{-n-\alpha q} dy\right)^{1/q} &\leqq M\left\{\int_0^\infty \left(\int_0^t s\|(\partial^2/\partial s^2)u\|_p ds\right)^q t^{-n-\alpha q} t^{n-1} dt\right\}^{1/q} \\
&\quad + M\sum_{i,j}\left(\int_0^\infty [t^3\|(\partial^3/\partial t\partial x_i\partial x_j)u\|_p]^q t^{-n-\alpha q} t^{n-1} dt\right)^{1/q} \\
&\quad + M\sum_{i,j}\left(\int_0^\infty [t^2\|(\partial^2/\partial x_i\partial x_j)u\|_p]^q t^{-n-\alpha q} t^{n-1} dt\right)^{1/q} \\
&\leqq M\left(\int_0^\infty [s^{2-\alpha}\|(\partial^2/\partial s^2)u\|_p]^q \, dt/t\right)^{1/q} \\
&\quad + M\sum_{i,j}\left(\int_0^\infty [t^{3-\alpha}\|(\partial^3/\partial t\partial x_i\partial x_j)u\|_p]^q \, dt/t\right)^{1/q} \\
&\quad + M\sum_{i,j}\left(\int_0^\infty [t^{2-\alpha}\|(\partial^2/\partial x_i\partial x_j)u\|_p]^q \, dt/t\right)^{1/q} \\
&\leqq M\left(\int_0^\infty [t^{2-\alpha}\|(\partial^2/\partial t^2)u\|_p]^q \, dt/t\right)^{1/q}.
\end{aligned}$$

THEOREM 5.3. *For $\alpha > 1$, $f \in \Lambda_\alpha^{p,q}$ if and only if $f \in L^p(\mathbf{R}^n)$ and $(\partial/\partial x_j)f \in \Lambda_{\alpha-1}^{p,q}$; moreover,*

$$\|f\|_{\Lambda_\alpha^{p,q}} \sim \|f\|_p + \sum_{i=1}^n \|(\partial/\partial x_i)f\|_{\Lambda_{\alpha-1}^{p,q}}.$$

PROOF. We give a proof only in the case $1 < \alpha < 2$. Suppose $f \in \Lambda_\alpha^{p,q}$. Since

$$(\partial^2/\partial t\partial x_i)u(x,t) = -\int_t^\infty (\partial^3/\partial s^2\partial x_i)u(x,s)\, ds,$$

we find by Hölder's inequality

$$\begin{aligned}
\|(\partial^2/\partial t\partial x_i)u(x,t)\|_p &\leqq \int_t^\infty \|(\partial^3/\partial s^2\partial x_i)u(x,s)\|_p \, ds \\
&\leqq Mt^{\alpha-2}\left(\int_t^\infty [s^{3-\alpha}\|(\partial^3/\partial s^2\partial x_i)u(x,s)\|_p]^q \, ds/s\right)^{1/q} \\
&\leqq MAt^{\alpha-2},
\end{aligned}$$

where $A = \left(\int_0^\infty [s^{2-\alpha}\|(\partial^2/\partial s^2)u(x,s)\|_p]^q \, ds/s\right)^{1/q}$. Hence it follows that for $0 < t_1 < t_2 < 1$,

$$\begin{aligned}
\|(\partial/\partial x_i)u(x,t_2) - (\partial/\partial x_i)u(x,t_1)\|_p &\leqq \int_{t_1}^{t_2} \|(\partial^2/\partial s\partial x_i)u(x,s)\|_p \, ds \\
&\leqq MA\int_{t_1}^{t_2} s^{\alpha-2}\, ds \leqq MA(t_2 - t_1),
\end{aligned}$$

which implies that $\{(\partial/\partial x_i)u(x,t)\}$ is a Cauchy net in $L^p(\mathbf{R}^n)$ as $t \to 0$. Consequently, we see that $(\partial/\partial x_i)f \in L^p(\mathbf{R}^n)$ and

$$\begin{aligned} &\left(\int_0^\infty [t^{1-(\alpha-1)}\|(\partial/\partial t)(P_t * (\partial f/\partial x_i))\|_p]^q \, dt/t\right)^{1/q} \\ = &\left(\int_0^\infty [t^{2-\alpha}\|(\partial^2/\partial t\partial x_i)u\|_p]^q \, dt/t\right)^{1/q}. \end{aligned}$$

The converse can be proved similarly.

COROLLARY 5.1. *Let $f \in L^p(\mathbf{R}^n)$. For $\alpha > 0$, take the nonnegative integer such that*

$$k < \alpha \leqq k+1.$$

Then $f \in \Lambda_\alpha^{p,q}$ if and only if $f \in W^{k,p}(\mathbf{R}^n)$ and

$$\left(\int_{\mathbf{R}^n} \frac{\|\nabla^k f(x+y) - 2\nabla^k f(x) + \nabla^k f(x-y)\|_p{}^q}{|y|^{n+(\alpha-k)q}} \, dy\right)^{1/q} < \infty.$$

LEMMA 5.3. *For $\alpha > 0$, $g_\alpha \in \Lambda_\alpha^{1,\infty}$.*

PROOF. We first consider the case $0 < \alpha < 1$. Note that

$$|(\partial/\partial x_i)g_\alpha(x)| \leqq M|x|^{\alpha-n-1},$$

so that

$$|g_\alpha(x+y) - g_\alpha(x)| \leqq M|y| \ |x|^{\alpha-n-1} \qquad \text{whenever } |x| > 2|y|.$$

Hence it follows that

$$\int_{\mathbf{R}^n - B(0,2|y|)} |g_\alpha(x+y) - g_\alpha(x)|dx \leqq M|y|^\alpha.$$

On the other hand,

$$\int_{B(0,2|y|)} |g_\alpha(x+y) - g_\alpha(x)|dx \leqq 2\int_{B(0,3|y|)} |g_\alpha(x)|dx \leqq M|y|^\alpha.$$

Now we obtain

$$\int |g_\alpha(x+y) - g_\alpha(x)|dx \leqq M|y|^\alpha,$$

which proves the case $0 < \alpha < 1$.

To show the general case, we write $\alpha = k\beta$, where k is a positive integer and $0 < \beta < 1$. Since $g_\alpha = g_\beta * g_\beta * \cdots * g_\beta$,

$$P_t g_\alpha = (P_{t_1} g_\beta) * \cdots * (P_{t_k} g_\beta)$$

for $t = t_1 + \cdots + t_k$, $t_i > 0$, so that

$$(\partial^k/\partial t)^k P_t g_\alpha = [(\partial/\partial t_1)(P_{t_1} g_\beta)] * \cdots * [(\partial/\partial t_k)(P_{t_k} g_\beta)].$$

If we take $t_i = t/k$, then it follows that

$$\|(\partial^k/\partial t^k)P_t g_\alpha\|_1 \leqq \|(\partial/\partial t_1)(P_{t_1}g_\beta)\|_1 \cdots \|(\partial/\partial t_k)(P_{t_k}g_\beta)\|_1 \leqq Mt^{-k(1-\beta)},$$

which implies that $g_\alpha \in \Lambda_\alpha^{1,\infty}$.

THEOREM 5.4. *For $\alpha > 0$ and $\beta \geqq 0$, g_β gives an isomorphism from $\Lambda_\alpha^{p,q}$ onto $\Lambda_{\alpha+\beta}^{p,q}$.*

PROOF. For $f \in \Lambda_\alpha^{p,q}$, set $u(x,t) = P_t f(x)$ and $U(x,t) = g_\beta * (P_t f)(x) = P_t(g_\beta f)(x)$. If k and ℓ are positive integers such that $k-1 \leqq \alpha < k$ and $\beta < \ell$, then

$$\begin{aligned} \|(\partial/\partial t)^{k+\ell}U\|_p &= \|[(\partial/\partial t_1)^\ell(P_{t_1}g_\beta)] * [(\partial/\partial t_2)^k(P_{t_2}f)]\|_p \\ &\leqq \|(\partial/\partial t_1)^\ell(P_{t_1}g_\beta)\|_1 \, \|(\partial/\partial t_2)^k(P_{t_2}f)\|_p \end{aligned}$$

for $t = t_1 + t_2$. By Lemma 5.3, we have for $t_1 = t_2 = t/2$,

$$\|(\partial/\partial t)^{k+\ell}U\|_p \leqq Mt^{\beta-\ell}\|(\partial/\partial t_2)^k u(x,t_2)\|_p,$$

so that

$$t^{k+\ell-(\alpha+\beta)}\|(\partial/\partial t)^{k+\ell}U\|_p \leqq Mt^{k-\alpha}\|(\partial/\partial t_2)^k u(x,t_2)\|_p.$$

Thus it follows that $g_\beta f \in \Lambda_{\alpha+\beta}^{p,q}$.

Next suppose $f \in \Lambda_{\alpha+2}^{p,q}$. By Lemma 5.2, $f \in \Lambda_\alpha^{p,q}$, and by Theorem 5.3, $\Delta f \in \Lambda_\alpha^{p,q}$, so that

$$(I - \Delta)f \in \Lambda_\alpha^{p,q}.$$

Noting that

$$f = g_2 * [(I - \Delta)f],$$

we see that g_2 maps $\Lambda_\alpha^{p,q}$ homeomorphically onto $\Lambda_{\alpha+2}^{p,q}$. Let $0 < \beta < 2$ and $f \in \Lambda_{\alpha+\beta}^{p,q}$. Then we find $g \in \Lambda_\alpha^{p,q}$ such that

$$g_2 * g = g_{2-\beta} * f \in \Lambda_{\alpha+2}^{p,q}.$$

Since $g_2 * g = g_{2-\beta} * [g_\beta * g]$, we see that

$$f = g_\beta * g,$$

which implies that $g_\beta(\Lambda_\alpha^{p,q}) = \Lambda_{\alpha+\beta}^{p,q}$.

The semigroup property of Bessel kernels proves the general case.

THEOREM 5.5. (1) *If $0 < \alpha_1 < \alpha_2 < \infty$, then $\Lambda_{\alpha_2}^{p,q_2} \subseteqq \Lambda_{\alpha_1}^{p,q_1}$ for any p, q_1, $q_2 \geqq 1$.*

(2) *If $1 \leqq q_1 < q_2 \leqq \infty$, then $\Lambda_\alpha^{p,q_1} \subseteqq \Lambda_\alpha^{p,q_2}$ for any $\alpha > 0$ and $p \geqq 1$.*

PROOF. First note that $\|(\partial/\partial t)^k u(x,t)\|_p$ is nonincreasing for any nonnegative integer k, because

$$\begin{aligned}\|(\partial/\partial t)^k u(x,t_1+t)\|_p &= \|P_{t_1}[(\partial/\partial t)^k u(x,t)]\|_p \\ &\leqq \|P_{t_1}\|_1 \|(\partial/\partial t)^k u(x,t)\|_p \leqq \|(\partial/\partial t)^k u(x,t)\|_p\end{aligned}$$

for any $t_1 > 0$ and $t > 0$. Suppose $f \in \Lambda_\alpha^{p,q}$ and k is a positive integer greater than α. Set

$$A = \left(\int_0^\infty [t^{k-\alpha}\|(\partial/\partial t)^k u\|_p]^q \, dt/t\right)^{1/q}.$$

Then we have

$$\begin{aligned}A^q &\geqq \int_{t/2}^t [s^{k-\alpha}\|(\partial/\partial s)^k u\|_p]^q \, ds/s \\ &\geqq \|(\partial/\partial t)^k u(x,t)\|_p]^q \int_{t/2}^t [s^{k-\alpha}]^q \, ds/s,\end{aligned}$$

so that

$$\|(\partial/\partial t)^k u(x,t)\|_p \leqq AMt^{\alpha-k}, \tag{5.3}$$

which implies that

$$\Lambda_\alpha^{p,q} \subseteqq \Lambda_\alpha^{p,\infty}.$$

Moreover, in case $q \leqq q_2 < \infty$, we have

$$\int_0^\infty [t^{k-\alpha}\|(\partial/\partial t)^k u\|_p]^{q_2} \, dt/t \leqq (AM)^{q_2-q}\int_0^\infty [t^{k-\alpha}\|(\partial/\partial t)^k u\|_p]^q \, dt/t < \infty,$$

so that

$$\Lambda_\alpha^{p,q} \subseteqq \Lambda_\alpha^{p,q_2},$$

which implies (2).

To show (1), we note from (5.3) that

$$\|(\partial/\partial t)^k u(x,t)\|_p \leqq Mt^{\alpha_2-k} \leqq Mt^{\alpha_1-k}$$

for $0 < t \leqq 1$, whenever $f \in \Lambda_{\alpha_2}^{p,\infty}$ and $\alpha_2 > \alpha_1 > 0$. For $t > 1$, we have

$$\|(\partial/\partial t)^k u(x,t)\|_p \leqq Mt^{-p'k}\|f\|_p \leqq M\|f\|_p t^{\alpha_1-k}.$$

Hence it follows that

$$\Lambda_{\alpha_2}^{p,\infty} \subseteqq \Lambda_{\alpha_1}^{p,\infty}.$$

If $q_1 < \infty$ and $f \in \Lambda_{\alpha_2}^{p,\infty}$, then

$$\begin{aligned}\int_0^\infty [t^{k-\alpha_1}\|(\partial/\partial t)^k u\|_p]^{q_1} \, dt/t &\leqq \int_0^1 [t^{k-\alpha_1}Mt^{\alpha_2-k}]^{q_1} \, dt/t \\ &\quad + \int_1^\infty [t^{k-\alpha_1}Mt^{-p'k}]^{q_1} \, dt/t < \infty,\end{aligned}$$

which shows that

$$\Lambda_{\alpha_2}^{p,\infty} \subseteqq \Lambda_{\alpha_1}^{p,q_1}.$$

Thus

$$\Lambda_{\alpha_2}^{p,q_2} \subseteqq \Lambda_{\alpha_2}^{p,\infty} \subseteqq \Lambda_{\alpha_1}^{p,q_1},$$

as required.

7.6 The relationships between L_α^p and $\Lambda_\alpha^{p,p}$

For $f \in L^p(\mathbf{R}^n)$, we set

$$g_1(f)(x) = \left(\int_0^\infty |(\partial/\partial t)u(x,t)|^2 t dt\right)^{1/2}$$

and

$$g(f)(x) = \left(\int_0^\infty |\nabla u(x,t)|^2 t dt\right)^{1/2},$$

where $\nabla = (\partial/\partial x_1, ..., \partial/\partial x_n, \partial/\partial t)$ and $u(x,t) = P_t * f(x)$.

LEMMA 6.1. *If $f \in L^2(\mathbf{R}^n)$, then*

$$\|g(f)\|_2 = 2^{-1/2}\|f\|_2$$

and

$$\|g_1(f)\|_2 = 2^{-1}\|f\|_2$$

PROOF. By Plancherel's formula, we have

$$\begin{aligned}
\|g(f)\|_2{}^2 &= \int_0^\infty \left(\int_{\mathbf{R}^n} |\nabla u(x,t)|^2 dx\right) t dt \\
&= \int_0^\infty \left(\int_{\mathbf{R}^n} [8\pi^2|y|^2|\hat{f}(y)|^2 e^{-4\pi t|y|}] dy\right) t dt \\
&= \int_{\mathbf{R}^n} 8\pi^2|y|^2|\hat{f}(y)|^2 \left(\int_0^\infty e^{-4\pi t|y|} t dt\right) dy \\
&= 2^{-1}\int_{\mathbf{R}^n} |\hat{f}(y)|^2 dy \\
&= 2^{-1}\int_{\mathbf{R}^n} |f(x)|^2 dy.
\end{aligned}$$

The remaining case can be treated similarly.

LEMMA 6.2. *If u_j are Poisson integrals of $f_j \in L^2(\mathbf{R}^n)$, respectively, then*

$$\int_{\mathbf{R}^n}\int_0^\infty (\partial/\partial t)u_1(x,t)\, \overline{(\partial/\partial t)u_2(x,t)}\, t dt dx = \frac{1}{4}\int_{\mathbf{R}^n} f_1(x)\overline{f_2(x)} dx.$$

PROOF. First we write

$$\begin{aligned}\|g_1(f_1+f_2)\|_2{}^2 &= \int_0^\infty \left(\int_{\mathbf{R}^n} |(\partial/\partial t)(u_1+u_2)|^2 dx\right) tdt \\ &= \int_0^\infty \left(\int_{\mathbf{R}^n} |(\partial/\partial t)u_1|^2 dx\right) tdt \\ &\quad + \int_0^\infty \left(\int_{\mathbf{R}^n} |(\partial/\partial t)u_2|^2 dx\right) tdt \\ &\quad + \int_0^\infty \left(\int_{\mathbf{R}^n} (\partial/\partial t)u_1\overline{(\partial/\partial t)u_2} dx\right) tdt \\ &\quad + \int_0^\infty \left(\int_{\mathbf{R}^n} \overline{(\partial/\partial t)u_1}(\partial/\partial t)u_2 dx\right) tdt.\end{aligned}$$

Hence, in view of Lemma 6.1, we have

$$4^{-1}\int (f_1\overline{f_2}+\overline{f_1}f_2)dx = \int_0^\infty \left(\int [(\partial/\partial t)u_1\overline{(\partial/\partial t)u_2}+\overline{(\partial/\partial t)u_1}(\partial/\partial t)u_2]dx\right) tdt,$$

which proves the required equality.

THEOREM 6.1. *If $f \in L^p(\mathbf{R}^n)$, then*

$$M^{-1}\|f\|_p \leqq \|g(f)\|_p \leqq M\|f\|_p.$$

PROOF. For $\varepsilon > 0$ and an integer j with $0 \leqq j \leqq n$, consider

$$K_\varepsilon(x) = ((\partial/\partial t)P_{t+\varepsilon}(x), (\partial/\partial x_1)P_{t+\varepsilon}(x), ..., (\partial/\partial x_n)P_{t+\varepsilon}(x)).$$

Denote by H_2 the family of all functions g for which

$$\int_0^\infty |g(t)|^2 tdt < \infty$$

and set ${H_2}^{n+1} = H_2 \times \cdots \times H_2$. Then note that

$$K_\varepsilon \in {H_2}^{n+1} \tag{6.1}$$

and

$$|(\partial/\partial x_j)K_\varepsilon(x)| \leqq M|x|^{-n-1}. \tag{6.2}$$

If we set

$$T_\varepsilon f(x) = \int K_\varepsilon(x-y)f(y)dy,$$

then

$$|T_\varepsilon f(x)| = \left(\int_0^\infty |\nabla u(x, t+\varepsilon)|^2 tdt\right)^{1/2} \leqq g(f)(x).$$

Further note that

$$\begin{aligned}|\hat{K}_\varepsilon(y)|^2 &= \int_0^\infty [|(\partial/\partial t)e^{-2\pi(t+\varepsilon)|y|}|^2 + |(2\pi y_1)e^{-2\pi(t+\varepsilon)|y|}|^2 + \cdots \\ &\quad +|(2\pi y_n)e^{-2\pi(t+\varepsilon)|y|}|^2]tdt \\ &\leqq 8\pi^2|y|^2\int_0^\infty e^{-4\pi(t+\varepsilon)|y|}tdt \leqq 2^{-1},\end{aligned}$$

or

$$|\hat{K}_\varepsilon(y)| \leqq 2^{-1/2}. \tag{6.3}$$

Thus, applying the Hilbert space version of singular integral operator theory, we find

$$\|T_\varepsilon f\|_p \leqq A_p\|f\|_p, \qquad 1 < p < \infty,$$

which gives, by letting $\varepsilon \to 0$,

$$\|g(f)\|_p \leqq A_p\|f\|_p.$$

Conversely, if u_j are Poisson integrals of f_j, respectively, then Lemma 6.2 gives

$$\int_{\mathbf{R}^n}\int_0^\infty (\partial/\partial t)u_1(x,t)\overline{(\partial/\partial t)u_2(x,t)}tdtdx = \frac{1}{4}\int_{\mathbf{R}^n} f_1(x)\overline{f_2(x)}dx,$$

which proves

$$\begin{aligned}\left|\int_{\mathbf{R}^n} f_1(x)\overline{f_2(x)}dx\right| &\leqq 4\int_{\mathbf{R}^n} g_1(f_1)(x)g_1(f_2)(x)dx \\ &\leqq 4\|g_1(f_1)\|_p\|g_1(f_2)\|_{p'} \\ &\leqq [4A_{p'}\|f_2\|_{p'}]\|g_1(f_1)\|_p.\end{aligned}$$

Hence it follows that

$$\|f_1\|_p \leqq 4A_{p'}\|g_1(f_1)\|_p.$$

THEOREM 6.2. *Let $\alpha > 0$ and $1 < p < \infty$. Then :*

(1) $L^p_\alpha \subseteqq \Lambda^{p,p}_\alpha$ *if* $p \geqq 2$.

(2) $L^p_\alpha \subseteqq \Lambda^{p,2}_\alpha$ *if* $p \leqq 2$.

(3) $\Lambda^{p,p}_\alpha \subseteqq L^p_\alpha$ *if* $p \leqq 2$.

(4) $\Lambda^{p,2}_\alpha \subseteqq L^p_\alpha$ *if* $p \geqq 2$.

PROOF. Since g_β gives isomorphisms of L^p_α as well as $\Lambda^{p,q}_\alpha$, we have only to deal the case $\alpha = 1$. Let $f \in L^p_1$ and denote by u the Poisson integral of f. Note that

$$[g((\partial/\partial x_j)f)(x)]^2 = \sum_{k=0}^n \int_0^\infty \left|\frac{\partial^2}{\partial x_j \partial x_k} u(x,t)\right|^2 tdt, \qquad x_0 = t,$$

so that

$$\int_0^\infty |\nabla^2 u(x,t)|^2 tdt = \sum_{j=1}^n [g((\partial/\partial x_j)f)(x)]^2$$

Since $\dfrac{\partial^2 u}{\partial t^2} = -\sum_{j=1}^n \dfrac{\partial^2 u}{\partial x_j^2}$ and

$$\sup_{t>0} t\left|\frac{\partial}{\partial x_j} u(x,t)\right| \leqq AMf(x),$$

we have

$$\int_0^\infty |(\partial/\partial t)^2 u(x,t)|^2 tdt \leqq A\sum_{k=1}^n [g((\partial/\partial x_j)f)(x)]^2$$

and

$$\sup_{t>0} t\left|\frac{\partial^2}{\partial t^2} u(x,t)\right| \leqq A\sum_{j=1}^n M((\partial/\partial x_j)f)(x).$$

Hence, if $p \geqq 2$, then

$$\int_0^\infty [t|(\partial/\partial t)^2 u|]^p dt/t \leqq A\left(\sum_{j=1}^n g((\partial/\partial x_j)f)(x)\right)^2 [M|\nabla f|]^{p-2},$$

so that Hölder's inequality gives

$$\left(\int_0^\infty [t\|(\partial/\partial t)^2 u\|_p]^p dt/t\right)^{1/p} \leqq A\|\nabla f\|_p.$$

Similarly, if $p \leqq 2$, then Minkowski's inequality for integral gives

$$\begin{aligned}\int_0^\infty [t\|(\partial/\partial t)^2 u\|_p]^2 dt/t \;&\leqq\; A\left\{\int \left(\int_0^\infty [t|(\partial/\partial t)^2 u|]^2 dt/t\right)^{p/2} dx\right\}^{2/p} \\ &\leqq\; A\left\{\int \left(\sum_{k=1}^n [g((\partial/\partial x_j)f)(x)]^2\right)^{p/2} dx\right\}^{2/p} \\ &\leqq\; A[\|\nabla f\|_p]^2.\end{aligned}$$

Thus (1) and (2) follow.

Conversely, if $p \geqq 2$, then Minkowski's inequality for integral also gives

$$\left\{\int \left(\int_0^\infty [t|(\partial/\partial t)^2 u|]^2 dt/t\right)^{p/2} dx\right\}^{2/p} \leqq \int_0^\infty [t\|(\partial/\partial t)^2 u\|_p]^2 dt/t,$$

which together with Theorem 6.1 yields

$$\|f\|_{L_1^p} \leqq A\|f\|_{\Lambda_1^{p,2}}, \qquad f \in L_1^p.$$

In case $1 < p \leqq 2$, from the first considerations and Theorem 6.1, we obtain

$$\begin{aligned}\sum_{j=1}^n \left\|\frac{\partial f}{\partial x_j}\right\|_p &\leqq A\left[\int\left\{\int_0^\infty \left(t\left|\frac{\partial^2 u}{\partial t^2}\right|\right)^2 \frac{dt}{t}\right\}^p dx\right]^{1/p} \\ &\leqq A\left\{\int_0^\infty \left(t\left\|\frac{\partial^2 u}{\partial t^2}\right\|_p\right)^p \frac{dt}{t}\right\}^{1/2} \left(\sum_{j=1}^n \left\|\frac{\partial f}{\partial x_j}\right\|_p\right)^{1-p/2},\end{aligned}$$

so that

$$\sum_{j=1}^n \left\|\frac{\partial f}{\partial x_j}\right\|_p \leqq A\left(\int_0^\infty [t\|(\partial/\partial t)^2 u\|_p]^p dt/t\right)^{1/p} \leqq A\|f\|_{\Lambda_1^{p,p}}.$$

If $f \in \Lambda_1^{p,p}$, then $u(x,\varepsilon) \in L_1^p$ for every $\varepsilon > 0$ and

$$\|u(x,\varepsilon)\|_{L_1^p} \leqq A\|u(x,\varepsilon)\|_{\Lambda_1^{p,p}} \leqq A\|f\|_{\Lambda_1^{p,p}}.$$

Since $u(x,\varepsilon) \to f$ in L^p as $\varepsilon \to 0$, we see that $f \in L_1^p$ and

$$\|f\|_{L_1^p} \leqq A\|f\|_{\Lambda_1^{p,p}},$$

as required.

7.7 Restriction and extension of Bessel potentials

First we show the restriction property of Bessel potentials.

THEOREM 7.1. *Let $\alpha > 0$ and $1 \leqq p \leqq \infty$. If $\beta = \alpha - 1/p > 0$, then $Ru \in \Lambda_\beta^{p,p}(\mathbf{R}^{n-1})$ for $u \in L_\alpha^p(\mathbf{R}^n)$ and*

$$\|Ru\|_{\Lambda_\beta^{p,p}(\mathbf{R}^{n-1})} \leqq M\|u\|_{L_\alpha^p(\mathbf{R}^n)}.$$

In this section, we write a point $x \in \mathbf{R}^n$ as

$$x = (s,\xi), \qquad s \in \mathbf{R}^1, \xi = (x_2, ..., x_n) \in \mathbf{R}^{n-1}.$$

To show Theorem 7.1, we first prepare the following result.

LEMMA 7.1. *If $g_\alpha^{(1)}$ denotes the Bessel kernel of order α in $\mathbf{R}^1$, then*

$$\int_{\mathbf{R}^{n-1}} g_\alpha(s,\xi)d\xi = Ag_\alpha^{(1)}(s).$$

In fact, note that

$$\begin{aligned}\int_{\mathbf{R}^{n-1}} g_\alpha(s,\xi)d\xi &= A_\alpha \int_0^\infty \left(\int_{\mathbf{R}^{n-1}} e^{-\pi(s^2+|\xi|^2)/\delta} d\xi\right) e^{-\delta/4\pi}\delta^{(\alpha-n)/2} d\delta/\delta \\ &= A_\alpha \int_0^\infty (Ae^{-\pi s^2/\delta}\delta^{(n-1)/2})e^{-\delta/4\pi}\delta^{(\alpha-n)/2} d\delta/\delta = A' g_\alpha^{(1)}(s).\end{aligned}$$

LEMMA 7.2. *Let $\alpha p > 1$. Then*

$$\left(\int_{\mathbf{R}^{n-1}} |g_\alpha f(0,\xi)|^p d\xi\right)^{1/p} \leqq M\|f\|_p.$$

PROOF. Set

$$K_1(s,\xi,\eta) = g_\alpha(s,\xi-\eta)[g_\alpha^{(1)}(s)]^{p'/p}$$

and

$$K_2(s,\xi,\eta) = \frac{g_\alpha(s,\xi-\eta)}{g_\alpha^{(1)}(s)}.$$

Then note by Lemma 7.1 that

$$\int_{\mathbf{R}^{n-1}} K_2(s,\xi,\eta)d\xi = A$$

and, since $(\alpha-1)p'+1>0$,

$$\int_{\mathbf{R}^1}\int_{\mathbf{R}^{n-1}} K_1(s,\xi,\eta)dsd\eta = A\int_{\mathbf{R}^1} [g_\alpha^{(1)}(s)]^{p'}ds = B < \infty.$$

Since $g_\alpha(s,\xi-\eta) = [K_1(s,\xi,\eta)]^{1/p'}[K_2(s,\xi,\eta)]^{1/p}$, for $\varphi \in L^{p'}(\mathbf{R}^{n-1})$ we have by Hölder's inequality and Lemma 7.1

$$\begin{aligned}&\int_{\mathbf{R}^{n-1}} [g_\alpha f(0,\xi)]\varphi(\xi)d\xi \\ =\ & \int_{\mathbf{R}^{n-1}} \left(\int_{\mathbf{R}^1}\int_{\mathbf{R}^{n-1}} g_\alpha(s,\xi-\eta)f(s,\eta)dsd\eta\right)\varphi(\xi)d\xi \\ =\ & \int_{\mathbf{R}^1}\int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}^{n-1}} \left([K_1(s,\xi,\eta)]^{1/p'}\varphi(\xi)\right)\left([K_2(s,\xi,\eta)]^{1/p}f(s,\eta)\right) dsd\eta d\xi \\ \leqq\ & \left(\int_{\mathbf{R}^1}\int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}^{n-1}} K_1(s,\xi,\eta)[\varphi(\xi)]^{p'}dsd\eta d\xi\right)^{1/p'} \\ &\qquad \times\left(\int_{\mathbf{R}^1}\int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}^{n-1}} K_2(s,\xi,\eta)[f(s,\eta)]^p dsd\eta d\xi\right)^{1/p} \\ \leqq\ & [B^{1/p'}\|\varphi\|_{p'}][A^{1/p}\|f\|_p],\end{aligned}$$

which shows that

$$\left(\int_{\mathbf{R}^{n-1}} |g_\alpha f(0,\xi)|^p d\xi\right)^{1/p} \leqq A^{1/p}B^{1/p'}\|f\|_p.$$

Lemma 7.3. *If* $0<\alpha<2$, *then*

$$\int_{\mathbf{R}^{n-1}} |g_\alpha(s,\xi+\eta)-g_\alpha(s,\xi)|d\xi \leqq M[|\eta|/|s|]^\varepsilon |s|^{\alpha-1}$$

whenever $2|\eta|>|s|$, *where* $\varepsilon=0$ *when* $\alpha<1$, $0<\varepsilon<1$ *when* $\alpha=1$ *and* $\varepsilon=1$ *when* $1<\alpha<2$.

Proof. If $\alpha<1$, then Lemma 7.1 gives

$$\int_{\mathbf{R}^{n-1}} |g_\alpha(s,\xi+\eta)-g_\alpha(s,\xi)|d\xi \leqq 2Ag_\alpha^{(1)}(s) \leqq M|s|^{\alpha-1}.$$

By mean value theorem, we have

$$\begin{aligned}\int_{\{\xi:|\xi|>2|\eta|\}} |g_\alpha(s,\xi+\eta)-g_\alpha(s,\xi)|d\xi &\leqq M|\eta|\int_{\{\xi:|\xi|>2|\eta|\}} |\xi|^{\alpha-n-1}d\xi \\ &\leqq M|\eta|^{\alpha-1} \leqq M[|\eta|/|s|]^\varepsilon|s|^{\alpha-1}\end{aligned}$$

when $2|\eta|>|s|$ and $\varepsilon>0$; M may depend on ε. On the other hand, we find

$$\begin{aligned}\int_{\{\xi:|\xi|<2|\eta|\}} |g_\alpha(s,\xi+\eta)-g_\alpha(s,\xi)|d\xi &\leqq 2\int_{\{\xi:|\xi|<3|\eta|\}} g_\alpha(s,\xi)d\xi \\ &\leqq M[|\eta|/|s|]^\varepsilon|s|^{\alpha-1}.\end{aligned}$$

Lemma 7.4. *Let* $0<\alpha<2$. *If* $2|\eta|<|s|$, *then*

$$\int_{\mathbf{R}^{n-1}} |g_\alpha(s,\xi+\eta)-2g_\alpha(s,\xi)+g_\alpha(s,\xi-\eta)|d\xi \leqq M|\eta|^2|s|^{\alpha-3}.$$

Proof. Since $2|\eta|<|s|$, we have for $|t|<1$,

$$|(s,\xi+t\eta)| \geqq |(s,\xi)|-|(0,\eta)| \geqq |(s,\xi)|/2.$$

Hence, by mean value theorem, we find that

$$|g_\alpha(s,\xi+\eta)-2g_\alpha(s,\xi)+g_\alpha(s,\xi-\eta)| \leqq M|\eta|^2|(s,\xi)|^{\alpha-n-2},$$

so that

$$\begin{aligned}&\int_{\mathbf{R}^{n-1}} |g_\alpha(s,\xi+\eta)-2g_\alpha(s,\xi)+g_\alpha(s,\xi-\eta)|d\xi \\ \leqq\ & M|\eta|^2\int_{\mathbf{R}^{n-1}} |(s,\xi)|^{\alpha-n-2}d\xi \leqq M|\eta|^2|s|^{\alpha-3}.\end{aligned}$$

Proof of Theorem 7.1. First consider the case $0<\alpha<2$. Let $u=g_\alpha f$ with $f\in L^p(\mathbf{R}^n)$, and write

$$u_s(\xi)=\int_{\mathbf{R}^{n-1}} g_\alpha(s,\xi-\eta)f(s,\eta)d\eta$$

and

$$u(\xi) \equiv u(0, \xi) = \int_{\mathbf{R}^1} u_s(\xi) ds.$$

Note that

$$\begin{aligned} \|u(\xi+\eta) - 2u(\xi) + u(\xi-\eta)\|_p &\leqq \int_{\mathbf{R}^1} \|u_s(\xi+\eta) - 2u_s(\xi) + u_s(\xi-\eta)\|_p ds \\ &= \int_{\{s:|s|<2|\eta|\}} \|u_s(\xi+\eta) - 2u_s(\xi) + u_s(\xi-\eta)\|_p ds \\ &\quad + \int_{\{s:|s|>2|\eta|\}} \|u_s(\xi+\eta) - 2u_s(\xi) + u_s(\xi-\eta)\|_p ds \\ &= I_1(\eta) + I_2(\eta). \end{aligned}$$

If $2|\xi| > |s|$, then Lemma 7.1 shows that

$$\int_{\mathbf{R}^{n-1}} |g_\alpha(s, \xi+\eta) - 2g_\alpha(s, \xi) + g_\alpha(s, \xi-\eta)| d\xi \leqq 3g_\alpha^{(1)}(s) \leqq M|s|^{\alpha-1},$$

so that Minkowski's inequality for integral gives

$$I_1(\eta) \leqq M \int_{\{s:|s|<2|\eta|\}} |s|^{\alpha-1} \|f(s, \cdot)\|_p ds.$$

Similarly, Lemma 7.4 gives

$$I_2(\eta) \leqq M|\eta|^2 \int_{\{s:|s|>2|\eta|\}} s^{\alpha-3} \|f(s, \cdot)\|_p ds.$$

Applying Hardy's inequality we obtain

$$\begin{aligned} \left(\int_{\mathbf{R}^{n-1}} \frac{[I_1(\eta)]^p}{|\eta|^{n-1+\beta p}} d\eta\right)^{1/p} &\leqq M \left\{\int_0^\infty r^{-\alpha p} \left(\int_0^{2r} s^{\alpha-1} \|f(s, \cdot)\|_p ds\right)^p dr\right\}^{1/p} \\ &\leqq M \left(\int_0^\infty s^{-\alpha p} [s^\alpha \|f(s, \cdot)\|_p]^p ds\right)^{1/p} \\ &= M \left(\int_0^\infty \|f(s, \cdot)\|_p{}^p ds\right)^{1/p} = M\|f\|_p. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \left(\int_{\mathbf{R}^{n-1}} \frac{[I_2(\eta)]^p}{|\eta|^{n-1+\beta p}} d\eta\right)^{1/p} &\leqq M \left\{\int_0^\infty r^{-\alpha p} \left(r^2 \int_{2r}^\infty s^{\alpha-3} \|f(s, \cdot)\|_p ds\right)^p dr\right\}^{1/p} \\ &\leqq M \left(\int_0^\infty s^{-\alpha p+2p} [s^{\alpha-2} \|f(s, \cdot)\|_p]^p ds\right)^{1/p} \\ &= M \left(\int_0^\infty \|f(s, \cdot)\|_p{}^p ds\right)^{1/p} = M\|f\|_p. \end{aligned}$$

Thus we find

$$\left(\int_{\mathbf{R}^{n-1}} \frac{[\|u(\xi+\eta) - 2u(\xi) + u(\xi-\eta)\|_p]^p}{|\eta|^{n-1+\beta p}} d\eta\right)^{1/p} \leqq M\|f\|_p.$$

Theorem 5.2 together with Lemma 7.2 now proves the case when $0 < \alpha < 2$.

If $m < \alpha \leqq m+1$, then, noting that

$$\nabla^m u = (\nabla^m g_\alpha) * f,$$

we apply the above considerations with g_α replaced by the partial derivatives of g_α of order m.

THEOREM 7.2. *Let $\alpha > 0$ and $1 \leqq p \leqq \infty$. If $\beta = \alpha - 1/p > 0$, then for each $f \in \Lambda_\beta^{p,p}(\mathbf{R}^{n-1})$ there exists an extension Ef in $L_\alpha^p(\mathbf{R}^n)$ of f to $\mathbf{R}^n$ such that*

$$\|Ef\|_{L_\alpha^p(\mathbf{R}^n)} \leqq M\|f\|_{\Lambda_\beta^{p,p}(\mathbf{R}^{n-1})}.$$

PROOF. Let $f \in \Lambda_\beta^{p,p}(\mathbf{R}^{n-1})$. Take a nonnegative function $\psi \in C_0^\infty(\mathbf{R}^{n-1})$ such that $\int \psi(\xi)d\xi = 1$ and $\psi(\xi) = 0$ for $|\xi| > 1$. Further take a nonnegative function $\lambda \in C_0^\infty(\mathbf{R}^1)$ such that $\lambda(t) = 1$ for $|t| < 1$. Now consider

$$\begin{aligned} u(t,\xi) &= \lambda(t) \int_{\mathbf{R}^{n-1}} f(\xi - |t|\eta)\psi(\eta)d\eta \\ &= \lambda(t)|t|^{1-n} \int_{\mathbf{R}^{n-1}} f(\eta)\psi((\xi-\eta)/|t|)d\eta. \end{aligned}$$

First we have by Minkowski's inequality for integral

$$\begin{aligned} \|u(t,\xi)\|_p &\leqq \int_{\mathbf{R}^{n-1}} \left(\int_{\mathbf{R}^n} |\lambda(t)f(\xi - |t|\eta)|^p dt d\xi\right)^{1/p} \psi(\eta)d\eta \\ &= \|\lambda\|_p \|f\|_p < \infty. \end{aligned}$$

Further we see that

$$\lim_{t\to 0} \|u(t,\cdot) - f(\cdot)\|_p = 0.$$

We first treat the case $0 < \alpha < 1$. In this case, we show that

$$U = g_{1-\alpha} * u \in W^{1,p}(\mathbf{R}^n).$$

Note that

$$\begin{aligned} (\partial/\partial\xi_j)u(t,\xi) &= \lambda(t)|t|^{-n} \int_{\mathbf{R}^{n-1}} f(\eta)(\partial/\partial\xi_j)[\psi((\xi-\eta)/|t|)]d\eta \\ &= \lambda(t)|t|^{-n} \int_{\mathbf{R}^{n-1}} [f(\xi-\eta) - f(\xi)](\partial\psi/\partial\xi_j)(\eta/|t|)d\eta, \end{aligned}$$

so that

$$|(\partial/\partial\xi_j)u(t,\xi)| \leqq M|t|^{-n} \int_{\{\eta:|\eta|<|t|\}} |f(\xi-\eta) - f(\xi)|d\eta.$$

We infer that

$$U_j \equiv g_{1-\alpha} * |(\partial u)/(\partial\xi_j)| \in L^p(\mathbf{R}^n). \tag{7.1}$$

In fact we have

$$\begin{aligned}|U_j(s,\zeta)| &\leqq M\int_{\mathbf{R}^n}|(t,\xi)|^{1-\alpha-n}|s-t|^{-n}\\ &\qquad\times\left(\int_{\{|\eta|<|s-t|\}}|f(\xi-\zeta-\eta)-f(\xi-\zeta)|d\eta\right)dtd\xi.\end{aligned}$$

Set

$$\omega(\eta)=\left(\int_{\mathbf{R}^{n-1}}|f(\xi-\eta)-f(\xi)|^p d\xi\right)^{1/p}$$

and

$$\Omega(r)=r^{1-\alpha-n}\int_{\{|\eta|<r\}}\omega(\eta)d\eta.$$

Then we see that

$$\begin{aligned}&\int_{\mathbf{R}^{n-1}}|U_j(s,\zeta)|^p d\zeta\leqq M\int_{\mathbf{R}^n}|(t,\xi)|^{1-\alpha-n}|s-t|^{-n}\\ &\qquad\times\left\{\int_{\{|\eta|<|s-t|\}}\left(\int_{\mathbf{R}^{n-1}}|f(\xi-\zeta-\eta)-f(\xi-\zeta)|^p d\zeta\right)^{1/p}d\eta\right\}dtd\xi\\ &= M\int_{\mathbf{R}^n}|(t,\xi)|^{1-\alpha-n}|s-t|^{\alpha-1}\Omega(|s-t|)dtd\xi\\ &= M\int_{\mathbf{R}^1}|t|^{-\alpha}|s-t|^{\alpha-1}\Omega(|s-t|)dt.\end{aligned}$$

If we set $K(s,t)=|t|^{-\alpha}|s-t|^{\alpha-1}$, then

$$\int_{\mathbf{R}}K(1,t)|t|^{-1/p}dt<\infty,$$

so that Lemma 2.1 in Chapter 4 gives

$$\int_{\mathbf{R}^1}\left(\int_{\mathbf{R}^{n-1}}|U_j(s,\zeta)|^p d\zeta\right)^{1/p}ds\leqq M\int_0^\infty\Omega(r)^p dr.$$

Further, noting that

$$\begin{aligned}\Omega(r) &= r^{1-\alpha-n}\int_0^r\left(\int_{S^{(n-1)}}\omega(r\Theta)d\Theta\right)r^{n-2}dr\\ &\leqq r^{1-\alpha-n}\int_0^r\left(\int_{S^{(n-1)}}\omega(r\Theta)^p d\Theta\right)^{1/p}r^{n-2}dr,\end{aligned}$$

we have by Hardy's inequality

$$\begin{aligned}\int_0^\infty\Omega(r)^p dr &\leqq M\int_0^\infty r^{p(1-\alpha-n)}\left(\int_{S^{(n-1)}}\omega(r\Theta)^p d\Theta\right)r^{p(n-2)+p}dr\\ &= M\int_{\mathbf{R}^{n-1}}|\eta|^{2-n-p\alpha}\omega(\eta)^p d\eta\\ &= M\int_{\mathbf{R}^{n-1}}|\eta|^{-p\beta-(n-1)}\left(\int_{\mathbf{R}^{n-1}}|f(\xi-\eta)-f(\xi)|^p d\xi\right)d\eta.\end{aligned}$$

On the other hand, note that

$$\begin{aligned}(\partial/\partial t)u(t,\xi) &= \lambda'(t)|t|^{-n+1}\int_{\mathbf{R}^{n-1}} f(\xi-\eta)\psi(\eta/|t|)d\eta \\ &\quad +\lambda(t)(1-n)|t|^{-n}(\mathrm{sgn}\ t)\int_{\mathbf{R}^{n-1}} f(\xi-\eta)\psi(\eta/|t|)d\eta \\ &\quad +\lambda(t)|t|^{1-n}\int_{\mathbf{R}^{n-1}} f(\xi-\eta)[(\nabla\psi)(\eta/|t|)\cdot(\eta/|t|^2)](-\mathrm{sgn}\ t)d\eta.\end{aligned}$$

Since $\int \eta_j(\partial\psi/\partial\eta_j)d\eta = -1$, we find that

$$\begin{aligned}(\partial/\partial t)u(t,\xi) &= \lambda'(t)|t|^{-n+1}\int_{\mathbf{R}^{n-1}} f(\xi-\eta)\psi(\eta/|t|)d\eta \\ &\quad +\lambda(t)(1-n)|t|^{-n}(\mathrm{sgn}\ t)\int_{\mathbf{R}^{n-1}} [f(\xi-\eta)-f(\xi)]\psi(\eta/|t|)d\eta \\ &\quad +\lambda(t)|t|^{-n}(\mathrm{sgn}\ t)\int_{\mathbf{R}^{n-1}} [f(\xi-\eta)-f(\xi)][(\nabla\psi)(\eta/|t|)\cdot(\eta/|t|)]d\eta \\ &= I_1+I_2+I_3.\end{aligned}$$

It suffices to note that

$$\begin{aligned}\|I_1\|_p &\leqq \int_{\mathbf{R}^{n-1}}\left(\int_{\mathbf{R}^n}|\lambda'(t)|^p f(\xi-|t|\eta)|^p dt d\xi\right)^{1/p}\psi(\eta)d\eta \\ &= \|\lambda'\|_p\|f\|_p<\infty\end{aligned}$$

and

$$|I_2|+|I_3| \leqq M|t|^{-n}\int_{\{\eta:|\eta|<|t|\}} |f(\xi-\eta)-f(\xi)|d\eta,$$

which can be evaluated as before.

In case $\alpha=1$, we have by Minkowski's inequality for integral and Hardy's inequality

$$\begin{aligned}&\int_{\mathbf{R}^n}\left(|t|^{-n}\int_{\{\eta:|\eta|<|t|\}} |f(\xi-\eta)-f(\xi)|d\eta\right)^p dt d\xi \\ \leqq\ & M\int_{\mathbf{R}^1}\left(|t|^{-n}\int_{\{\eta:|\eta|<|t|\}}\omega(\eta)d\eta\right)^p dt \\ =\ & M\int_{\mathbf{R}^1}|t|^{-pn}\left\{\int_0^{|t|}\left(\int_{S^{n-1}}\omega(r\Theta)d\Theta\right)r^{n-2}dr\right\}^p dt \\ \leqq\ & M\int_0^\infty r^{-pn}\left\{\left(\int_{S^{n-1}}\omega(r\Theta)d\Theta\right)r^{n-1}\right\}^p dr \\ =\ & M\int_{\mathbf{R}^{n-1}}\frac{\|f(\xi-\eta)-f(\xi)\|_p{}^p}{|\eta|^{(n-1)+\beta p}}\,d\eta.\end{aligned}$$

The general case will be left to the reader as an exercise.

Chapter 8

Boundary limits

In this chapter we study various boundary limits for functions on the half space $\mathbf{H}$. In fact, fine limits, perpendicular limits, radial limits and curvilinear limits are considered for Green potentials and Beppo Levi functions. The existence of tangential limits is discussed for polyharmonic functions together with monotone functions.

8.1 Boundary limits for Green potentials

We recall (Theorem 4.3 in Chapter 3) that a nonnegative superharmonic function s on the half space $\mathbf{H} = \{x = (x_1, ..., x_n) : x_1 > 0\}$ is represented as

$$s(x) = ax_1 + \int_{\mathbf{H}} G(x,y)d\mu(y) + \int_{\partial\mathbf{H}} P(x,y)d\nu(y),$$

where a is a nonnegative constant and μ, ν are measures on $\mathbf{H}$, $\partial\mathbf{H}$, respectively. To obtain general results, we consider Green's function G_α of order α, which is given by

$$G_\alpha(x,y) = \begin{cases} |x-y|^{\alpha-n} - |\overline{x}-y|^{\alpha-n} & \text{when } 0 < \alpha < n, \\ \\ \log(|\overline{x}-y|/|x-y|) & \text{when } \alpha = n, \end{cases}$$

where $\overline{x} = (-x_1, x_2, ..., x_n)$ for $x = (x_1, x_2, ..., x_n)$.

LEMMA 1.1. *For $x = (x_1, ..., x_n) \in \mathbf{H}$ and $y = (y_1, ..., y_n) \in \mathbf{H}$, in case $\alpha < n$,*

$$M^{-1} \frac{x_1 y_1}{|x-y|^{n-\alpha}|\overline{x}-y|^2} \leqq G_\alpha(x,y) \leqq M \frac{x_1 y_1}{|x-y|^{n-\alpha}|\overline{x}-y|^2};$$

in case $\alpha = n$,

$$M^{-1} \frac{x_1 y_1}{|\overline{x}-y|^2} \leqq G_n(x,y) \leqq M \frac{x_1 y_1}{|x-y|^2}.$$

PROOF. Set $t = |\overline{x}-y|/|x-y| > 1$ for $x = (x_1, ..., x_n) \in \mathbf{H}$ and $y = (y_1, ..., y_n) \in \mathbf{H}$. In case $\alpha < n$,

$$G_\alpha(x,y) = |\overline{x}-y|^{\alpha-n}(t^{n-\alpha} - 1).$$

Hence it suffices to note that

$$M^{-1}t^{n-\alpha-1} \leqq \frac{t^{n-\alpha}-1}{t-1} \leqq Mt^{n-\alpha-1}$$

and

$$t-1 = \frac{t^2-1}{t+1} = \frac{4x_1y_1|\overline{x}-y|}{|x-y|+|\overline{x}-y|}.$$

The case $\alpha = n$ can be treated similarly.

COROLLARY 1.1. *For a measure μ on $\mathbf{H}$, $G_\alpha\mu \not\equiv \infty$ if and only if*

$$\int_{\mathbf{H}} (1+|y|)^{\alpha-n-2} y_1 d\mu(y) < \infty. \tag{1.1}$$

Another application of Lemma 1.1 shows that

$$k_\alpha(x,y) = x_1^{-1}y_1^{-1}G_\alpha(x,y)$$

is extended to be a continuous function on $\overline{\mathbf{H}} \times \overline{\mathbf{H}}$ in the extended sense; in fact, if x and y are in $\partial\mathbf{H}$, then

$$k_\alpha(x,y) = a_\alpha|x-y|^{\alpha-n-2}, \qquad a_\alpha = \begin{cases} 2(n-\alpha) & \text{when } \alpha < n, \\ 2 & \text{when } \alpha = n. \end{cases}$$

THEOREM 1.1. *If s is nonnegative and superharmonic in $\mathbf{H}$, then there exist $a > 0$ and a measure λ on $\partial\mathbf{H}$ such that*

$$s(x) = ax_1 + x_1 \int_{\overline{\mathbf{H}}} k_2(x,y)d\lambda(y) \qquad \textit{for } x \in \mathbf{H}.$$

We know that $x_1k_2(x,y)$ is the Poisson kernel for $\mathbf{H}$, and the Poisson integral

$$P_{x_1}(x',\lambda) = x_1 \int_{\partial\mathbf{H}} k_2(x,y)d\lambda(y)$$

has a nontangential limit at almost every boundary point (see Theorem 3.1 in Chapter 3). Thus we are concerned with the boundary limits of Green potentials

$$G_\alpha\mu(x) = \int_{\mathbf{H}} G_\alpha(x,y)d\mu(y) = x_1\int_{\mathbf{H}} k_\alpha(x,y)d\lambda(y),$$

where $d\lambda(y) = y_1 d\mu(y)$.

We first consider perpendicular limits of Green potentials $G_\alpha\mu$. For this purpose, let $e = (1,0,...,0)$.

THEOREM 1.2. *Let μ be a measure on $\mathbf{H}$ such that*

$$\int_{\mathbf{H}} {y_1}^{\beta} d\mu(y) < \infty \tag{1.2}$$

for $0 \leqq \beta \leqq 1$. Then, for $0 < \gamma \leqq 1$, there exists a set $E \subsetneqq \partial\mathbf{H}$ such that

(i) *if* $\xi \in \partial \mathbf{H} - E$, *then*

$$\lim_{r \to 0} r^{-\gamma} G_\alpha \mu(\xi + re) = 0 \text{ when } \gamma < 1,$$

$$\lim_{r \to 0} r^{-1} G_\alpha \mu(\xi + re) = \int_{\mathbf{H}} k_\alpha(\xi, y) y_1 d\mu(y) \text{ when } \gamma = 1;$$

(ii) $H_{n-\alpha+\beta+\gamma}(E) = 0$.

To show this, we prepare the following results; recall that

$$\Gamma(\xi, a) = \{x = (x_1, ..., x_n) : |x - \xi| < a x_1\}.$$

LEMMA 1.2. *Let* $\gamma < 1$ *and* μ *be a measure on* $\mathbf{H}$ *such that* $G_\alpha \mu \not\equiv \infty$. *Then, for* $\xi \in \partial \mathbf{H}$, *the following are equivalent.*

(i) $\displaystyle \lim_{x \to \xi, x \in \Gamma(\xi, a)} x_1^{-\gamma} \int_{\mathbf{H} - B(x, x_1/2)} G_\alpha(x, y) d\mu(y) = 0$ *for any* $a > 0$;

(ii) $\displaystyle \lim_{r \to 0} r^{-\gamma+1} \int_{\mathbf{H} \cap B(\xi, 1)} (r + |\xi - y|)^{\alpha - n - 2} y_1 d\mu(y) = 0;$

(iii) $\displaystyle \lim_{r \to 0} r^{\alpha - n - \gamma - 1} \int_{B_+(\xi, r)} y_1 d\mu(y) = 0,$

where $B_+(\xi, r) = \mathbf{H} \cap B(\xi, r)$.

PROOF. First note that

$$M^{-1} x_1 < |x - \xi| < M x_1 \qquad \text{whenever } x \in \Gamma(\xi, a).$$

Hence we have by Lemma 1.1,

$$M^{-1} x_1^{-\gamma+1} (x_1 + |\xi - y|)^{\alpha - n - 2} y_1 \leqq x_n^{-\gamma} G_\alpha(x, y) \leqq M x_1^{-\gamma+1} (x_1 + |\xi - y|)^{\alpha - n - 2} y_1$$

whenever $y \in \mathbf{H} - B(x, x_1/2)$. Consequently, we see that (i) is equivalent to (ii). Clearly, (ii) implies (iii).

For $r > 0$, set

$$\mu(r) = r^{\alpha - n - \gamma - 1} \int_{B_+(\xi, r)} y_1 d\mu(y).$$

If (iii) holds, then $\lim_{r \to 0} \mu(r) = 0$. For $0 < r < \delta$, find

$$\begin{aligned} & r^{-\gamma+1} \int_{B_+(\xi, \delta)} (r + |\xi - y|)^{\alpha - n - 2} y_1 d\mu(y) \\ = \; & r^{-\gamma+1} (r + \delta)^{\alpha - n - 2} \int_{B_+(\xi, \delta)} y_1 d\mu(y) \\ & + r^{-\gamma+1} \int_0^\delta \left(\int_{B_+(\xi, t)} y_1 d\mu(y) \right) d(-(r + t)^{\alpha - n - 2}) \\ \leqq \; & \mu(\delta) + \mu(\delta) r^{-\gamma+1} \int_0^\delta t^{n - \alpha + \gamma + 1} d(-(r + t)^{\alpha - n - 2}) \leqq M \mu(\delta). \end{aligned}$$

This shows that (iii) implies (ii).

LEMMA 1.3. *Let $\ell \geqq 0$ and μ be a finite measure on* $\mathbf{H}$. *If we set*

$$A_\ell = \left\{ \xi \in \partial\mathbf{H} : \limsup_{r \to 0} \ r^{-\ell}\mu(B_+(\xi, r)) > 0 \right\},$$

then $H_\ell(A_\ell) = 0$.

PROOF. For each $j > 0$, we show that $H_\ell(E_j) = 0$, where

$$E_j = \left\{ \xi \in \partial\mathbf{H} \cap B(0, j) : \limsup_{r \to 0} \ r^{-\ell}\mu(B_+(\xi, r)) > 1/j \right\}.$$

If $\delta > 0$ and $\xi \in E_j$, then there exists $r = r(\xi)$ such that $0 < r < \delta$ and

$$r^{-\ell}\mu(B_+(\xi, r)) > 1/j.$$

By a covering lemma (see Theorem 10.1 in Chapter 1), there exists a mutually disjoint family $\{B_i\}$ such that $B_i = B(\xi_i, r(\xi_i))$ and $\bigcup_i 5B_i \supseteqq E_j$. Then we have

$$H_\ell^{(5\delta)}(E_j) \leqq \sum_i [5r(\xi_i)]^\ell \leqq 5^\ell j \sum_i \mu(\mathbf{H} \cap B_i) \leqq 5^\ell j \mu(F_\delta),$$

where $F_\delta = \{x \in \mathbf{H} : x_1 < \delta\}$. This shows by letting $\delta \to 0$ that $H_\ell(E_j) = 0$, as required.

PROOF OF THEOREM 1.2. Write

$$u_1(x) = \int_{B(x, x_1/2)} G_\alpha(x, y) d\mu(y)$$

and

$$u_2(x) = \int_{\mathbf{H} - B(x, x_1/2)} G_\alpha(x, y) d\mu(y).$$

First we see by using Lebesgue's dominated convergence theorem that

$$\lim_{x \to \xi, x \in \Gamma(\xi, a)} x_1^{-1} u_2(x) = \int_{\mathbf{H}} k_\alpha(x, y) y_1 d\mu(y);$$

if the right-hand side is not finite, then Fatou's lemma gives the equality. In case $\gamma < 1$, we see from Lemma 1.2 that

$$\lim_{x \to \xi, x \in \Gamma(\xi, a)} x_1^{-\gamma} u_2(x) = 0$$

holds for every $\xi \in \partial\mathbf{H} - A_{n-\alpha+\gamma+\beta}$, since

$$r^{\alpha-n-\gamma-1} \int_{B_+(\xi, r)} y_1 d\mu(y) \leqq r^{\alpha-n-\gamma-\beta} \int_{B_+(\xi, r)} y_1^\beta d\mu(y).$$

Further note that

$$u_1(x) \leqq \int_{B(x,x_1/2)} |x-y|^{\alpha-n} d\mu(y).$$

For a sequence $\{a_j\}$ of positive numbers, consider the sets

$$E_j = \{x \in \mathbf{H} : 2^{-j} \leqq x_1 < 2^{-j+1}, x_1^{-\gamma} u_1(x) > a_j^{-1}\}$$

and

$$E = \bigcap_{k=1}^{\infty} \left(\bigcup_{j=k}^{\infty} E_j \right)^*,$$

where A^* denotes the projection of A to the hyperplane $\partial\mathbf{H}$. If $\alpha < n$ and $\xi \in E_j^*$, then there exists $x \in E_j$ for which

$$\begin{aligned} a_j^{-1} &< x_1^{-\gamma} u_1(x) < x_1^{-\gamma} \int_{B(x,x_1/2)} |x-y|^{\alpha-n} d\mu(y) \\ &\leqq 2^{n-\alpha} x_1^{\alpha-n-\gamma} \mu(B(x,x_1/2)) + x_1^{-\gamma} \int_0^{x_1/2} \mu(B(x,r)) d(-r^{\alpha-n}), \end{aligned}$$

so that there exists $r = r(x)$ such that $0 < r \leqq x_1/2$ and

$$\mu(B(x,r)) > 2^{\gamma}[(n-\alpha)/\gamma + 1]^{-1} a_j^{-1} r^{n-\alpha+\gamma}.$$

The case $\alpha = n$ can be treated similarly. By a covering lemma (see Theorem 10.1 in Chapter 1), we can choose a family $\{B_i\}$ such that $B_i = B(x_i, r(x_i))$ and $\bigcup_i 5B_i \supseteqq E_j$. Then we have

$$\begin{aligned} H_{n-\alpha+\beta+\gamma}^{(5\cdot 2^{-j})}(E_j^*) &\leqq \sum_i [5r(\xi_i)]^{n-\alpha+\beta+\gamma} \\ &\leqq M a_j 2^{-j\beta} \sum_i \mu(B_i) \\ &\leqq M a_j 2^{-j\beta} \mu(A_j), \end{aligned}$$

where $A_j = \{x : 2^{-j-1} < x_1 < 2^{-j+2}\}$. Now we choose $\{a_j\}$ so that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_j a_j \int_{A_j} y_1^{\beta} d\mu(y) < \infty.$$

Then it follows that

$$H_{n-\alpha+\beta+\gamma}(E) = 0$$

and

$$\limsup_{r\to 0} r^{-\gamma} u_1(\xi + re) \leqq \limsup_{j\to\infty} a_j^{-1} = 0$$

whenever $\xi \in \partial\mathbf{H} - E$. Thus Theorem 1.2 is obtained.

Next we treat the case $\gamma \leqq 0$. In the logarithmic case $\alpha = n$, we had better consider the capacity :

$$C_n(E) = \inf\ \lambda(\mathbf{H}), \qquad E \subseteqq \mathbf{H},$$

where the infimum is taken over all measures λ on $\mathbf{H}$ such that

$$\int_{\mathbf{H}} G_n(x,y)d\lambda(y) \geqq 1 \qquad \text{for all } x \in E.$$

THEOREM 1.3. *Let μ be a measure on $\mathbf{H}$ satisfying* (1.2) *for $\beta \leqq 1$. If $-\beta \leqq \gamma \leqq 0$, then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $C_{\alpha-\beta-\gamma}(E) = 0$ and*

$$\lim_{r \to 0} r^{-\gamma} G_\alpha \mu(\xi + re) = 0 \qquad \textit{for every } \xi \in \partial\mathbf{H} - E.$$

PROOF. Write $G_\alpha\mu(x) = u_1(x) + u_2(x)$ as above. Then u_2 has a nontangential limit at every $\xi \in \partial\mathbf{H} - A_{n-\alpha+\beta+\gamma}$. Since $H_{n-\alpha+\beta+\gamma}(A_{n-\alpha+\beta+\gamma}) = 0$, we have

$$C_{\alpha-\beta-\gamma}(A_{n-\alpha+\beta+\gamma}) = 0$$

in view of Theorem 7.5 in Chapter 2. To deal with u_1, consider

$$E_j = \{x \in \mathbf{H} : 2^{-j} \leqq x_1 < 2^{-j+1}, x_1^{-\gamma}u_1(x) > a_j^{-1}\}$$

and

$$E = \bigcap_{k=1}^{\infty} \left(\bigcup_{j=k}^{\infty} E_j \right)^* .$$

In case $\alpha < n$, noting that

$$\begin{aligned} a_j^{-1} &< x_1^{-\gamma}u_1(x) < x_1^{-\gamma} \int_{B(x,x_1/2)} |x-y|^{\alpha-n} d\mu(y) \\ &\leqq 2^{-\gamma} \int_{A_j} |x-y|^{\alpha-\beta-\gamma-n} y_1^\beta d\mu(y) \end{aligned}$$

for $x \in E_j$, we have

$$C_{\alpha-\beta-\gamma}(E_j) \leqq 2^{-\gamma} a_j \int_{A_j} y_1^\beta d\mu(y).$$

The case $\alpha = n$ can be treated similarly. Hence it follows that

$$\begin{aligned} C_{\alpha-\beta-\gamma}(\bigcup_{j=k}^{\infty} E_j^*) &\leqq \sum_{j=k}^{\infty} C_{\alpha-\beta-\gamma}(E_j^*) \\ &\leqq \sum_{j=k}^{\infty} C_{\alpha-\beta-\gamma}(E_j) \\ &\leqq \sum_{j=k}^{\infty} 2^{-\gamma} a_j \int_{A_j} y_1^\beta d\mu(y). \end{aligned}$$

Now we choose $\{a_j\}$ so that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_j a_j \int_{A_j} y_1^\beta d\mu(y) < \infty.$$

Then we find

$$C_{\alpha-\beta-\gamma}(E) = 0$$

and

$$\limsup_{r\to 0} r^{-\gamma} u_1(\xi + re) \leqq \limsup_{j\to\infty} a_j^{-1} = 0$$

whenever $\xi \in \partial\mathbf{H} - E$. Thus Theorem 1.3 is obtained.

Here we discuss the existence of global fine limits.

THEOREM 1.4. *Let μ be a measure on $\mathbf{H}$ satisfying* (1.2) *for $\alpha - n - 1 < \beta \leqq 1$. Then there exists a set $E \subseteqq \mathbf{H}$ such that*

$$\lim_{x_1\to 0, x\in\mathbf{H}-E} x_1^{n-\alpha+\beta} G_\alpha\mu(x) = 0$$

and

$$\sum_{j=1}^{\infty} 2^{j(n-\alpha)} C_\alpha(E_j) < \infty, \tag{1.3}$$

where $E_j = \{x \in E : 2^{-j} \leqq x_1 < 2^{-j+1}\}$.

PROOF. Write $G_\alpha\mu(x) = u_1(x) + u_2(x)$ as above. Since $x_1^{n-\alpha+\beta} G_\alpha(x,y) \leqq M y_1^\beta$ for $y \in \mathbf{H} - B(x, x_1/2)$, we see that $u_2(x)$ tends to zero as $x_1 \to 0$, with the aid of Lebesgue's dominated convergence theorem. Consider

$$E_j = \{x \in \mathbf{H} : 2^{-j} \leqq x_1 < 2^{-j+1}, x_1^{n-\alpha+\beta} u_1(x) > a_j^{-1}\}$$

as before. Here we show only the case $\alpha = n$. Since

$$a_j^{-1} < x_1^\beta u_1(x) < M \int_{B(x,x_1/2)} \log(|\overline{x} - y|/|x - y|) y_1^\beta d\mu(y)$$

for $x \in E_j$, we find

$$C_n(E_j) \leqq M a_j \int_{A_j} y_1^\beta d\mu(y).$$

Now we may choose $\{a_j\}$ so that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_j a_j \int_{A_j} y_1^\beta d\mu(y) < \infty.$$

PROPOSITION 1.1. *Let $\alpha = n$ and $E \subseteqq \mathbf{H}$. Then* (1.3) *holds if and only if*

$$C_n(E) < \infty. \tag{1.4}$$

PROOF. Since C_n is countably subadditive, (1.3) implies (1.4). Conversely, suppose $C_n(E) < \infty$ and take a nonnegative finite measure λ such that

$$G_n\lambda(x) \geqq 1 \qquad \text{whenever } x \in E.$$

Write $G_n\lambda(x) = u_1(x) + u_2(x)$ as before. Since $G_n(x,y) \leqq Mx_1y_1|x-y|^{-2} \leqq M$ whenever $y \in \mathbf{H} - B(x, x_1/2)$ and $\lambda(\mathbf{H}) < \infty$, we see that

$$\lim_{x_1 \to 0} u_2(x) = 0.$$

This implies that if j is large enough, then

$$u_1(x) \geqq 1/2 \qquad \text{whenever } x \in E_j,$$

so that

$$C_n(E_j) \leqq 2\lambda(A_j).$$

Noting that $\sum_j \lambda(A_j) \leqq 3\lambda(\mathbf{H}) < \infty$, we insist that (1.4) implies (1.3).

PROPOSITION 1.2. *Let $n-1 < \alpha \leqq n$ and C, $C = \{C(r) : 0 \leqq r < 1\}$, be a curve in $\mathbf{H}$ tending to a boundary point. If μ is a measure on $\mathbf{H}$ satisfying* (1.1), *then*

$$\liminf_{r \to 1}\ [C_1(r)]^{n-\alpha+1} G_\alpha\mu(C(r)) = 0,$$

where $C(r) = (C_1(r), ..., C_n(r))$.

PROOF. Letting

$$L_j = \{(x_1, 0) : 2^{-j} \leqq x_1 < 2^{-j+1}\},$$

we note that

$$C_\alpha(L_j) = 2^{-j(n-\alpha)} C_\alpha(L_0)$$

and

$$C_\alpha(L_0) > 0 \qquad \text{when } n-1 < \alpha \leqq n.$$

This implies that the line segment $(0,1] \times \{0\}$ fails to satisfy (1.3) when $n-1 < \alpha \leqq n$. To show the second assertion, we have only to consider the potential

$$u(y) = \int_0^1 G_\alpha((t,0), y)dt.$$

Since u is bounded on $\mathbf{H}$ when $\alpha > n-1$, letting $u \leqq M$ on $\mathbf{H}$, we find

$$\mu(\mathbf{H}) \geqq M^{-1} \int_{\mathbf{H}} u(y)d\mu(y) = M^{-1} \int_0^1 G_\alpha\mu((t,0))dt \geqq M^{-1}$$

for all measures μ on $\mathbf{H}$ such that $G_\alpha \mu(x) \geqq 1$ on L_0. This shows that $C_\alpha(L_0) \geqq M^{-1} > 0$. For $x \in \mathbf{H}$, denote the projection of x to the x_1-axis. For a measure μ on $\mathbf{H}$, define a measure μ^* on x_1-axis by setting

$$\mu^*(A^*) = \mu(\{x : (x_1, 0) \in A^*\}).$$

Then we infer that

$$C_\alpha(E^*) \leqq C_\alpha(E),$$

which follows from the fact that

$$G_\alpha \mu(x) \leqq \int_{\mathbf{H}} G_\alpha(x^*, y^*) d\mu(y) \leqq G_\alpha \mu^*(x^*),$$

where $x^* = (x_1, 0)$ for $x = (x_1, x')$. Consequently,

$$C_\alpha(L_j) \leqq C_\alpha(C_j) \qquad \text{with } C_j = \{x \in C : 2^{-j} \leqq x_1 < 2^{-j+1}\}.$$

Now the required assertion follows from Theorem 1.4.

We show the radial limit result for Green potentials.

THEOREM 1.5. *Let μ be a measure on $\mathbf{H}$ satisfying* (1.2) *for $\beta \leqq 1$. Then, for $\alpha - \beta - n \leqq \gamma < 1$, there exists a set $E \subseteq \mathbf{H}$ such that*

(i) *if $\xi \in \partial\mathbf{H} - E$, then $\lim_{r \to 0} r^{-\gamma} G_\alpha \mu(\xi + r\zeta) = 0$ for every $\zeta \in \mathbf{H} \cap \mathbf{S} - E_\xi$ with $C_\alpha(E_\xi) = 0$.*

(ii) $H_{n-\alpha+\beta+\gamma}(E) = 0$.

To show this, we need the following result.

LEMMA 1.4. *Let $n - \alpha + \beta + \gamma \geqq 0$. For a measure μ on $\mathbf{H}$ satisfying* (1.2), *set*

$$F_{\beta'} = \left\{ \xi \in \partial\mathbf{H} : \int_{B_+(\xi,1)} |\xi - y|^{\alpha - n - \beta' - \gamma} y_1^{\beta'} d\mu(y) = \infty \right\},$$

If $\beta' > \beta$, then

$$H_{n-\alpha+\beta+\gamma}(F_{\beta'}) = 0.$$

PROOF. Suppose $H_{n-\alpha+\beta+\gamma}(F_{\beta'}) > 0$. Then, in view of Frostman's theorem, we can find a measure ν on $\mathbf{R}^n$ such that S_ν is a compact subset of $F_{\beta'}$, $\nu(\mathbf{R}^n) > 0$ and

$$\nu(B(x,r)) \leqq r^{n-\alpha+\beta+\gamma} \qquad \text{for all } B(x,r).$$

Letting $S_\nu \subseteqq B(0,a)$, we note that

$$\begin{aligned}
\infty &= \int \left(\int_{B_+(\xi,1)} |\xi - y|^{\alpha-n-\beta'-\gamma} y_1^{\beta'} d\mu(y) \right) d\nu(\xi) \\
&= \int_{B_+(0,a+1)} \left(\int |\xi - y|^{\alpha-n-\beta'-\gamma} d\nu(\xi) \right) y_1^{\beta'} d\mu(y) \\
&\leqq \int_{B_+(0,a+1)} \left(\int_0^\infty r^{n-\alpha+\beta+\gamma} d(-(r^2+y_1^2)^{(\alpha-n-\beta'-\gamma)/2}) \right) y_1^{\beta'} d\mu(y) \\
&\leqq M \int_{B_+(0,a+1)} y_1^{\beta} d\mu(y) < \infty,
\end{aligned}$$

which is a contradiction.

Proof of Theorem 1.5. Write $G_\alpha\mu(x) = u_1(x) + u_2(x)$ as in the proof of Theorem 1.2. First we see that

$$x_1^{-\gamma} u_1(x) \leqq M \int_{B(x,x_1/2)} |x-y|^{\alpha-n} y_1^{-\gamma} d\mu(y)$$

and

$$\lim_{x\to\xi, x\in\Gamma(\xi,a)} x_1^{-\gamma} u_2(x) = 0$$

for every $\xi \in \partial\mathbf{H} - A_{n-\alpha+\gamma+\beta}$, on account of Lemma 1.2. If $\xi \in \partial\mathbf{H} - F_{\beta'}$, then

$$\int_{\Gamma(\xi,a)\cap B(\xi,1)} |\xi - y|^{\alpha-n} y_1^{-\gamma} d\mu(y) < \infty$$

for every $a > 0$. Hence, in view of the radial limit theorem for Riesz potentials (see Theorem 8.1 in Chapter 2), we can find a set $E_\xi \subseteqq \mathbf{H}\cap\mathbf{S}$ such that $C_\alpha(E_\xi) = 0$ and

$$\lim_{r\to 0} r^{-\gamma} u_1(\xi + r\zeta) = 0$$

for every $\zeta \in \mathbf{H}\cap\mathbf{S} - E_\xi$. Thus, since $H_{n-\alpha+\gamma+\beta}(A_{n-\alpha+\gamma+\beta} \cup F_{\beta'}) = 0$, Theorem 1.5 is established.

Proposition 1.3. *Let s be a nonnegative superharmonic function on $\mathbf{H}$. Then, for $1-n \leqq \gamma \leqq 1$, there exists a set $E \subseteqq \partial\mathbf{H}$ with the following properties:*

(i) *if $\xi \in \partial\mathbf{H} - E$, then there exists s_ξ such that $\lim_{r\to 0} r^{-\gamma} s(\xi + r\zeta) = s_\xi$ for every $\zeta \in \mathbf{H}\cap\mathbf{S} - E_\xi$ with $C_2(E_\xi) = 0$.*

(ii) *$H_{n-1+\gamma}(E) = 0$.*

We say that a set E in $\mathbf{H}$ is minimally α-semithin at the origin if

$$\lim_{r\to 0} r^{\alpha-n-2} C_{k_\alpha}(E \cap B(0,r)) = 0.$$

Note that E is minimally α-semithin at the origin if and only if

$$\lim_{j\to\infty} 2^{-j(\alpha-n-2)} C_{k_\alpha}(E_j) = 0, \tag{1.5}$$

where $E_j = \{x \in E : 2^{-j} \leqq |x| < 2^{-j+1}\}$. In fact, it suffices to note that

$$\begin{aligned} 2^{-j(\alpha-n-2)} C_{k_\alpha}(E \cap B(0, 2^{-j+1})) &\leqq 2^{-j(\alpha-n-2)} \sum_{i=j}^{\infty} C_{k_\alpha}(E_i) \\ &\leqq 2^{-j(\alpha-n-2)} \sum_{i=j}^{\infty} [a_j 2^{i(\alpha-n-2)}] \\ &= a_j/[1-2^{\alpha-n-2}], \end{aligned}$$

where $a_j = \sup \{2^{-i(\alpha-n-2)} C_{k_\alpha}(E_i) : i \geqq j\}$.

We also say that a function u on $\mathbf{H}$ has minimally α-semifine limit zero at the origin if there exists a set $E \subseteqq \mathbf{H}$ such that E is minimally α-semithin at 0 and

$$\lim_{x\to 0, x\in \mathbf{H}-E} u(x) = 0.$$

PROPOSITION 1.4. *If E is minimally α-semithin at 0, then $E \cap \Gamma(0,a)$ is α-semithin at 0, that is,*

$$\lim_{r\to 0} r^{\alpha-n} C_\alpha(E \cap \Gamma(0,a) \cap B(0,2r) - B(0,r)) = 0.$$

REMARK 1.1. In case $\alpha < n$, E is α-semithin at 0 if and only if

$$\lim_{r\to 0} r^{\alpha-n} C_\alpha(E \cap B(0,r)) = 0. \tag{1.6}$$

PROOF OF PROPOSITION 1.4. For $x \in \mathbf{H}$ and a measure λ on $\overline{\mathbf{H}}$, write

$$\begin{aligned} k_\alpha(x,\lambda) &= \int_{B(x,x_1/2)} k_\alpha(x,y)\,d\lambda(y) \\ &\quad + \int_{\overline{\mathbf{H}}-B(x,x_1/2)} k_\alpha(x,y)\,d\lambda(y) = u_1(x) + u_2(x). \end{aligned}$$

Note first that if $x \in \Gamma(0,a)$, then

$$u_2(x) \leqq M x_1^{\alpha-n-2} \lambda(\overline{\mathbf{H}}) \leqq M|x|^{\alpha-n-2} \lambda(\overline{\mathbf{H}}).$$

Hence, if $2^{-j(\alpha-n-2)} C_{k_\alpha}(E_j) < \varepsilon < (2M)^{-1}$, then we can find λ such that $\lambda(\overline{\mathbf{H}}) < 2^{j(\alpha-n-2)}\varepsilon$ and $k_\alpha(x,\lambda) \geqq 1$ for all $x \in E_j$. In this case, $u_2(x) < 2^{-1}$ for all $x \in E_j$, so that $u_1(x) > 2^{-1}$ for all $x \in E_j$. Since

$$u_1(x) \leqq M x_1^{-2} \int_{B_j} |x-y|^{\alpha-n} d\lambda(y),$$

where $B_j = \{x : 2^{-j-1} < |x| < 2^{-j+2}\}$, it follows that

$$C_\alpha(E_j \cap \Gamma(0,a)) \leqq M2^{2j}\lambda(B_j) \leqq M2^{j(\alpha-n)}\varepsilon,$$

so that

$$2^{j(n-\alpha)}C_\alpha(E_j \cap \Gamma(0,a)) \leqq M\varepsilon.$$

This shows that $E \cap \Gamma(0,a)$ is α-semithin at 0 if E is minimally α-semithin at 0.

THEOREM 1.6. *Let μ be a measure on* $\mathbf{H}$ *satisfying* (1.1), *and* $-1 \leqq \gamma < 1$. *Then the following are equivalent.*

(i) *If* $1 \leqq p < n/(n-\alpha+\gamma+1)$, *then*

$$\lim_{r\to 0} r^{-n}\int_{B_+(0,r)}[x_1|x|^{-1-\gamma}G_\alpha\mu(x)]^p dx = 0.$$

(ii) *There exists a sequence* $\{x^{(j)}\}$ *tending to* 0 *such that* $\{x^{(j)}\} \subseteqq \Gamma(0,a)$ *for some* $a > 0$, $|x^{(j)}| < b|x^{(j+1)}|$ *for some* $b > 1$ *and*

$$\lim_{j\to\infty} |x^{(j)}|^{-\gamma}G_\alpha\mu(x^{(j)}) = 0.$$

(iii) $x_1^{-1}|x|^{1-\gamma}G_\alpha\mu(x)$ *has minimally α-semifine limit zero at* 0.

(iv) $\lim_{r\to 0} r^{\alpha-\gamma-1-n}\int_{B_+(0,r)} y_1 d\mu(y) = 0.$

PROOF. If (i) holds, then, setting $A(r) = B(0,2r) - B(0,r)$, we have

$$\lim_{r\to 0}\frac{1}{|A(r)\cap\Gamma(0,1)|}\int_{A(r)\cap\Gamma(0,1)}[x_1|x|^{-1-\gamma}G_\alpha\mu(x)]^p dx = 0.$$

Hence (ii) holds for $x^{(j)} \in A(2^{-j}) \cap \Gamma(0,1)$ such that

$$[x_1^{(j)}|x^{(j)}|^{-1-\gamma}G_\alpha\mu(x^{(j)})]^p \leqq \frac{1}{|A(r)\cap\Gamma(0,1)|}\int_{A(r)\cap\Gamma(0,1)}[x_1|x|^{-1-\gamma}G_\alpha\mu(x)]^p dx.$$

For $x \in \Gamma(0,a)$, note from Lemma 1.1 that

$$\begin{aligned}|x|^{-\gamma}G_\alpha\mu(x) &\geqq M|x|^{-\gamma}\int_{B_+(0,|x|)}[x_1y_1|\overline{x}-y|^{\alpha-n-2}]d\mu(y)\\ &\geqq M|x|^{-\gamma+1+\alpha-n-2}\int_{B_+(0,|x|)} y_1 d\mu(y).\end{aligned}$$

If (ii) holds for $\{x^{(j)}\}$, then

$$\lim_{j\to\infty} |x^{(j)}|^{\alpha-\gamma-1-n}\int_{B_+(0,|x^{(j)}|)} y_1 d\mu(y) = 0;$$

since $(0,|x^{(1)}|] \subseteqq \bigcup_j (b^{-1}|x^{(j)}|,|x^{(j)}|]$, (iv) is satisfied.

For $r>0$, write

$$\begin{aligned} G_\alpha\mu(x) &= \int_{B_+(0,2r)} G_\alpha(x,y)d\mu(y) \\ &\quad + \int_{\mathbf{H}-B(0,2r)} G_\alpha(x,y)d\mu(y) = u_r(x)+v_r(x). \end{aligned}$$

By Minkowski's inequality for integral we have

$$\begin{aligned} &\left(r^{-n}\int_{B_+(0,r)}[x_1|x|^{-1-\gamma}u_r(x)]^p dx\right)^{1/p} \\ \leqq\ & \int_{B_+(0,2r)}\left(r^{-n}\int_{B_+(0,r)}[x_1|x|^{-1-\gamma}G_\alpha(x,y)]^p dx\right)^{1/p} d\mu(y) \\ \leqq\ & M\int_{B_+(0,2r)}\left(r^{-n}\int_{B(0,r)}|x-y|^{p(\alpha-n+1-\gamma)}dx\right)^{1/p} y_1 d\mu(y) \\ \leqq\ & Mr^{\alpha-n+1-\gamma}\int_{B_+(0,2r)} y_1 d\mu(y). \end{aligned}$$

If (iv) holds, then

$$\lim_{r\to 0}\frac{1}{|B_+(0,r)|}\int_{B_+(0,r)}[x_1|x|^{-1-\gamma}u_r(x)]^p dx = 0.$$

On the other hand, note for $x\in B_+(0,r)$

$$x_1|x|^{-1-\gamma}v_r(x) \leqq Mr^{1-\gamma}\int_{\mathbf{H}-B(0,2r)}|y|^{\alpha-n-2}y_1 d\mu(y).$$

For $\delta>0$, set

$$\varepsilon(\delta) = \sup_{0<r<\delta} r^{\alpha-\gamma-1-n}\int_{B_+(0,r)} y_1 d\mu(y).$$

Then it follows that

$$\begin{aligned} &\limsup_{r\to 0}\ r^{1-\gamma}\int_{\mathbf{H}-B(0,2r)}|y|^{\alpha-n-2}y_1 d\mu(y) \\ =\ & \limsup_{r\to 0}\ r^{1-\gamma}\int_{B_+(0,\delta)-B(0,2r)}|y|^{\alpha-n-2}y_1 d\mu(y) \\ \leqq\ & \varepsilon(\delta)\limsup_{r\to 0}\ r^{1-\gamma}\int_{2r}^{\delta} t^{-\alpha+\gamma+1+n}d(-t^{\alpha-n-2}) \leqq M\varepsilon(\delta). \end{aligned}$$

Hence, if (iv) holds, then

$$\lim_{r\to 0}\frac{1}{|B_+(0,r)|}\int_{B_+(0,r)}[x_1|x|^{-1-\gamma}v_r(x)]^p dx = 0.$$

Thus (iv) implies (i); then (i), (ii), (iv) are equivalent.

Next we show that (iv) implies (iii). For this purpose, write $G_\alpha \mu(x) = u_{|x|}(x) + v_{|x|}(x)$ as above. Then we see that

$$x_1^{-1}|x|^{1-\gamma} v_{|x|}(x) \leqq M|x|^{1-\gamma} \int_{\mathbf{H}-B(0,2|x|)} |y|^{\alpha-n-2} y_1 d\mu(y),$$

which tends to zero, as noted above. For a sequence $\{a_j\}$ of positive numbers, consider the sets

$$E_j = \{x \in \mathbf{H} : 2^{-j} \leqq |x| < 2^{-j+1}, x_1^{-1}|x|^{1-\gamma} u_{|x|}(x) \geqq a_j^{-1}\}.$$

Noting that

$$\begin{aligned} x_1^{-1}|x|^{1-\gamma} u_{|x|}(x) &\leqq |x|^{1-\gamma} \int_{B_+(0,2|x|)} k_\alpha(x,y) y_1 d\mu(y) \\ &\leqq M 2^{-j(1-\gamma)} \int_{B_+(0,2|x|)} k_\alpha(x,y) y_1 d\mu(y), \end{aligned}$$

we have

$$C_{k_\alpha}(E_j) \leqq M a_j 2^{-j(1-\gamma)} \int_{B_+(0,2^{-j+2})} y_1 d\mu(y),$$

so that

$$2^{j(n-\alpha+2)} C_{k_\alpha}(E_j) \leqq M a_j 2^{j(n-\alpha+1+\gamma)} \int_{B_+(0,2^{-j+2})} y_1 d\mu(y).$$

Now choose $\{a_j\}$ such that $\lim_{j\to\infty} a_j = \infty$ and

$$\lim_{j\to\infty} a_j 2^{j(n-\alpha+1+\gamma)} \int_{B_+(0,2^{-j+2})} y_1 d\mu(y) = 0.$$

Then, setting $E = \bigcup_j E_j$, we see that E is minimally α-semithin at 0 and

$$\lim_{x\to 0, x\in \mathbf{H}-E} x_1^{-1}|x|^{1-\gamma} u_{|x|}(x) = 0.$$

Hence (iv) implies (iii).

If (iii) holds, then

$$\lim_{x\to 0, x\in\Gamma(0,a)-E} |x|^{-\gamma} G_\alpha \mu(x) = 0$$

for a set E which is minimally α-semithin at 0. In view of Proposition 1.4,

$$\lim_{r\to 0} \frac{C_\alpha(E \cap \Gamma(0,a) \cap B(0,2r) - B(0,r))}{C_\alpha(\Gamma(0,a) \cap B(0,2r) - B(0,r))} = 0,$$

so that we can find $\{x^{(j)}\}$ such that $x^{(j)} \in \Gamma(0,a) \cap [B(0,2^{-j+1}) - B(0,2^{-j})] - E$ for each j. Then (ii) holds for this sequence $\{x^{(j)}\}$, and hence (iii) implies (ii). Thus (i) - (iv) are all equivalent.

For $0 \leqq \beta \leqq 1$, $x \in \mathbf{H}$ and $y \in \mathbf{H}$, define

$$k_{\alpha,\beta}(x,y) = x_1^{-1} y_1^{-\beta} G_\alpha(x,y);$$

as above, $k_{\alpha,1} = k_\alpha$. We consider the capacity $C_{k_{\alpha,\beta}}$ defined by

$$C_{k_{\alpha,\beta}}(E) = \inf \ \lambda(\overline{\mathbf{H}}), \qquad E \subseteqq \mathbf{H},$$

where the supremum is taken over all measures λ such that

$$k_{\alpha,\beta}(x,\lambda) = \int_{\overline{\mathbf{H}}} k_{\alpha,\beta}(x,y) d\lambda(y) \geqq 1 \qquad \text{for every } x \in E.$$

Finally we study the fine limits at infinity for Green potentials in $\mathbf{H}$.

THEOREM 1.7. *Let μ be a measure on $\mathbf{H}$ such that*

$$\int_{\mathbf{H}} (1+|y|)^{\alpha-n-\gamma-1} y_1 d\mu(y) < \infty \tag{1.7}$$

for $\gamma \leqq 1$. If $0 \leqq \beta \leqq 1$, then there exists a set $E \subseteqq \mathbf{H}$ such that

$$\lim_{|x|\to\infty, x\in\mathbf{H}-E} x_n^{-\beta} |x|^{\beta-\gamma} G_\alpha\mu(x) = 0$$

and

$$\sum_{j=1}^{\infty} 2^{-j(n-\alpha+\beta+1)} C_{k_{\alpha,\beta}}(E_j) < \infty. \tag{1.8}$$

REMARK 1.2. If (1.8) holds for $\alpha = 2$, then E is called β-rarefied at ∞; if (1.8) holds for $\beta = 1$, then E is called minimally α-thin at ∞.

PROOF OF THEOREM 1.7. For $x \in \mathbf{H}$, write

$$\begin{aligned} G_\alpha\mu(x) &= \int_{B_+(x,|x|/2)} G_\alpha(x,y) d\mu(y) \\ &\quad + \int_{\mathbf{H}-B(x,|x|/2)} G_\alpha(x,y) d\mu(y) = u_1(x) + u_2(x). \end{aligned}$$

If $x \in \mathbf{H}$ and $y \in \mathbf{H} - B(x,|x|/2)$, then

$$x_1^{-\beta}|x|^{\beta-\gamma} G_\alpha(x,y) \leqq M x_1^{1-\beta} |x|^{\beta-\gamma} |x-y|^{\alpha-n-2} y_1 \leqq M(|x|+|y|)^{\alpha-n-\gamma-1} y_1.$$

Hence we apply Lebesgue's dominated convergence theorem to find

$$\lim_{|x|\to\infty, x\in\mathbf{H}} x_n^{-\beta} |x|^{\beta-\gamma} u_2(x) = 0.$$

By assumption (1.7) there exists a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j 2^{j(n-\alpha+1+\gamma)} \int_{\mathbf{H}\cap B_{-j}} y_1 d\mu(y) < \infty; \tag{1.9}$$

recall that $B_{-j} = \{x : 2^{j-1} < |x| < 2^{j+2}\}$. Consider the sets

$$E_j = \{x \in \mathbf{H} : 2^j \leqq |x| < 2^{j+1}, x_1^{-\beta}|x|^{\beta-\gamma}u_1(x) \geqq a_j^{-1}\}.$$

Noting that

$$x_1^{-\beta}|x|^{\beta-\gamma}u_1(x) \leqq |x|^{\beta-\gamma} \int_{\mathbf{H}\cap B_{-j}} k_{\alpha,\beta}(x,y)y_1 d\mu(y),$$

we have

$$C_{k_{\alpha,\beta}}(E_j) \leqq M a_j 2^{j(\beta-\gamma)} \int_{\mathbf{H}\cap B_{-j}} y_1 d\mu(y).$$

This together with (1.9) yields (1.8).

Corollary 1.2. *Let μ be a measure on $\mathbf{H}$ satisfying (1.7) for $\gamma \leqq 1$. If $0 \leqq \beta \leqq 1$, then there exists a set $E \subseteqq \mathbf{H} \cap \mathbf{S}$ such that $C_\alpha(E) = 0$ and*

$$\lim_{r\to\infty} r^{-\gamma} G_\alpha \mu(rx) = 0 \qquad \text{for every } x \in \mathbf{H} \cap \mathbf{S} - E.$$

To show this, it suffices to note that

$$M^{-1}C_{k_{\alpha,\beta}}(E_j) \leqq 2^{j(\beta+1)}C_\alpha(E_j) \leqq MC_{k_{\alpha,\beta}}(E_j) \qquad \text{whenever } E \subseteqq \Gamma(0,a).$$

8.2 Boundary limits for BLD functions

In this section we are concerned with various boundary limits of BLD functions u on $\mathbf{H}$ satisfying

$$\int_G |\nabla^m u(x)|^p {x_1}^\beta dx < \infty \qquad \text{for any bounded open set } G \subseteqq \mathbf{H}. \tag{2.1}$$

First we show the following.

Lemma 2.1. *For $\beta > -1$, let $\gamma = \beta - p$ when $\beta > p-1$ and $-1 < \gamma < \beta$ when $\beta \leqq p-1$. If $u \in BL_1(L^p_{loc}(\mathbf{H}))$ satisfies*

$$\int_G |\nabla u(x)|^p x_1^\beta dx < \infty \qquad \textit{for any bounded open set } G \subseteqq \mathbf{H}, \tag{2.2}$$

then

$$\int_G |u(x)|^p x_1^\gamma dx < \infty \qquad \textit{for any bounded open set } G \subsetneqq \mathbf{H}. \tag{2.3}$$

PROOF. For $a > 0$, find $b > a > 0$ such that

$$\int_{\{x':|x'|<a\}} |u(b,x')|^p dx' < \infty,$$

where $x = (x_1, x_2, ..., x_n) = (x_1, x')$. Note that

$$u(x) = -\int_{x_1}^b (\partial u)/(\partial x_1)(t,x')dt + u(b,x')$$

for almost every $x' \in \mathbf{R}^{n-1}$. Letting $\varepsilon = \beta - \gamma - 1$ when $\beta \leqq p-1$ and $p-1 < \varepsilon < \beta$ when $\beta > p-1$, we have by Hölder's inequality

$$|u(x)| \leqq \left(\int_{x_1}^b |\nabla u(t,x')|^p t^\varepsilon dt\right)^{1/p} \left(\int_{x_1}^b t^{-p'\varepsilon/p} dt\right)^{1/p'} + |u(b,x')|.$$

Hence it follows from Fubini's theorem that

$$\int_0^b |u(x_1,x')|^p x_1^\gamma dx_1 \leqq M\int_0^b |\nabla u(t,x')|^p t^\beta dt + M|u(b,x')|^p,$$

which together with (2.2) gives

$$\int_{B_+(0,a)} |u(x)|^p x_1^\gamma dx < \infty;$$

recall that $B_+(0,a) = \mathbf{H} \cap B(0,a)$. Thus (2.3) follows.

COROLLARY 2.1. *Let $-1 < \beta < mp-1$, ℓ be a nonnegative integer such that $\ell p - 1 \leqq \beta < (\ell+1)p - 1$, and $u \in BL_m(L^p_{loc}(\mathbf{H}))$ satisfy (2.1). If $\ell p - 1 < \beta$, then*

$$\int_G |\nabla^{m-\ell} u(x)|^p x_1^{\beta-\ell p} dx < \infty \qquad \textit{for any bounded open set } G \subsetneqq \mathbf{H}; \tag{2.4}$$

if $\ell p - 1 = \beta$ and $-1 < \gamma < p-1$, then

$$\int_G |\nabla^{m-\ell} u(x)|^p x_1^\gamma dx < \infty \qquad \textit{for any bounded open set } G \subsetneqq \mathbf{H}. \tag{2.5}$$

THEOREM 2.1. *If $u \in BL_m(L^p_{loc}(\mathbf{H}))$ satisfies (2.1) for $-1 < \beta < p-1$, then there exists an extension Eu to the whole space $\mathbf{R}^n$ satisfying*

$$\int_G |\nabla^m Eu(x)|^p |x_1|^\beta dx < \infty \qquad \textit{for any bounded open set } G \subsetneqq \mathbf{R}^n. \tag{2.1$'$}$$

PROOF. Let $\lambda_1,...,\lambda_{m+1}$ be a unique solution of the linear system :

$$\left\{\begin{array}{ccccccccc} \lambda_1 & + & \lambda_2 & + & \cdots & + & \lambda_{m+1} & = & 1, \\ (-1)\,\lambda_1 & + & (-2)\,\lambda_2 & + & \cdots & + & (-m-1)\,\lambda_{m+1} & = & 1, \\ & & & & \vdots & & & & \vdots \\ (-1)^m\lambda_1 & + & (-2)^m\lambda_2 & + & \cdots & + & (-m-1)^m\lambda_{m+1} & = & 1. \end{array}\right.$$

For functions u on $\mathbf{H}$, we define

$$Eu(x) = \begin{cases} u(x) & \text{if } x_1 > 0, \\ \displaystyle\sum_{j=1}^{m+1} \lambda_j u(-jx_1, x') & \text{if } x_1 < 0 \end{cases}$$

and for each multi-index $\mu = (\mu_1, ..., \mu_n)$,

$$E_\mu u(x) = \begin{cases} u(x) & \text{if } x_1 > 0, \\ \displaystyle\sum_{j=1}^{m+1} (-j)^{\mu_1} \lambda_j u(-jx_1, x') & \text{if } x_1 < 0. \end{cases}$$

If u is ACL on $\mathbf{H}$, then Eu is defined to be ACL on $\mathbf{R}^n$ and

$$D^\mu(Eu) = E_\mu(D^\mu u) \qquad \text{for } |\mu| = 1.$$

Thus, if u is a BLD function on $\mathbf{H}$ satisfying (2.1), then Theorem 5.2 in Chapter 6 implies that Eu is defined to be BLD on $\mathbf{R}^n$ satisfying (2.1$'$) and

$$D^\mu(Eu) = E_\mu(D^\mu u) \qquad \text{for } |\mu| \leqq m.$$

Recall that $k_\lambda(x) = x^\lambda/|x|^n$ and

$$k_{\lambda,\ell}(x,y) = \begin{cases} k_\lambda(x-y) & \text{if } |y| \leqq 1, \\ k_\lambda(x-y) - \displaystyle\sum_{|\mu| \leqq \ell} (1/\mu!) x^\mu (D^\mu k_\lambda)(-y) & \text{if } |y| > 1. \end{cases}$$

In view of Theorem 1.3 in Chapter 6, we have the following representation of u.

THEOREM 2.2. *Let $u \in BL_m(L^p_{loc}(\mathbf{H}))$ satisfy (2.1) for $-1 < \beta < p-1$, and ℓ be the integer such that $\ell \leqq m - (n+\beta)/p < \ell + 1$. Then there exists a polynomial P of degree at most $m-1$ such that*

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int k_{\lambda,\ell}(x,y) D^\lambda Eu(y)dy + P(x)$$

for almost every $x \in \mathbf{H}$.

Theorem 2.2 can be proved for Eu in the same manner as Theorem 1.3 in Chapter 6, if one notes that $k_{\lambda,\ell}(x,y) \leqq M|x|^{\ell+1}|y|^{m-n-\ell-1}$ for $|y| > \max\{2|x|, 1\}$ and

$$\int (1+|y|)^{m-n-\ell-1}|D^\lambda Eu(y)|dy \leqq \left(\int (1+|y|)^{p'(m-n-\ell-1)}|y_1|^{-p'\beta/p}dy\right)^{1/p'} \times \left(\int |D^\lambda Eu(y)|^p|y_1|^\beta dy\right)^{1/p} < \infty$$

when $\beta < p-1$ and $m-(n+\beta)/p < \ell+1$.

Note here that for $R > 0$,

$$\int k_{\lambda,\ell}(x,y)D^\lambda Eu(y)dy = \int_{B(0,R)} k_\lambda(x-y)D^\lambda Eu(y)dy + \text{ a continuous function on } B(0,R).$$

Hence, when we want to study the existence of boundary limits for the functions u on $\mathbf{H}$, we have only to deal with the functions of the form :

$$U_\lambda f(x) = \int \frac{(x-y)^\lambda}{|x-y|^n} f(y)dy,$$

where $|\lambda| = m$, and f is a function on $\mathbf{R}^n$ which vanishes outside a compact set and satisfies

$$\int_{\mathbf{R}^n} |f(y)|^p|y_1|^\beta dy < \infty. \tag{2.6}$$

THEOREM 2.3. *Let* $|\lambda| = m$ *and* $-1 < \beta < p-1$. *For* $\ell \leqq m-(n+\beta)/p < \ell+1$ *and a function* f *satisfying* (2.6), *set*

$$U_{\lambda,\ell}f(x) = \int k_{\lambda,\ell}(x,y)f(y)dy.$$

Then

$$\int_{\mathbf{R}^n} |\nabla^m U_{\lambda,\ell}f(x)|^p|x_1|^\beta dx \leqq M \int_{\mathbf{R}^n} |f(y)|^p|y_1|^\beta dy.$$

PROOF. For $\varepsilon > 0$, set

$$K_\varepsilon(x) = x^\lambda(|x|^2+\varepsilon^2)^{-n/2}$$

and

$$K_{\varepsilon,\ell}(x,y) = \begin{cases} K_\varepsilon(x-y) & \text{if } |y| \leqq 1, \\ K_\varepsilon(x-y) - \sum_{|\mu|\leqq\ell} (1/\mu!)x^\mu(D^\mu K_\varepsilon)(-y) & \text{if } |y| > 1. \end{cases}$$

Consider the function

$$F_{\varepsilon,\ell}(x) = \int K_{\varepsilon,\ell}(x,y) f(y)|y_1|^{\beta/p} dy.$$

Then $F_{\varepsilon,\ell}$ is infinitely differentiable for $x \in \mathbf{R}^n$. Since $\nabla^m K_{\varepsilon,\ell} = \nabla^m K_\varepsilon$, with the aid of singular integral theory, we see that

$$\int_{\mathbf{R}^n} |\nabla^m F_{\varepsilon,\ell}(x)|^p dx \leqq M \int_{\mathbf{R}^n} |f(y)|^p |y_1|^\beta dy.$$

Note also that $K_{\varepsilon,\ell} * f$ is infinitely differentiable on $\mathbf{R}^n$ and

$$\begin{aligned} ||x_1|^{\beta/p} D^\mu (K_{\varepsilon,\ell} * f)(x) - D^\mu F_{\varepsilon,\ell}(x)| &\leqq M \int_{\mathbf{R}^n} \frac{||x_1|^{\beta/p} - |y_1|^{\beta/p}|}{|x-y|^n} |f(y)| dy \\ &= M \int_{-\infty}^{\infty} K(x_1,y_1) g(x_1,x',y_1) dy_1, \end{aligned}$$

where $|\mu| = m$,

$$K(x_1,y_1) = \frac{|1 - (|x_1|/|y_1|)^{\beta/p}|}{|x_1 - y_1|}$$

and

$$g(x_1,x',y_1) = \int_{\mathbf{R}^{n-1}} \frac{|x_1 - y_1|}{[|x'-y'|^2 + (x_1-y_1)^2]^{n/2}} |f(y)||y_1|^{\beta/p} dy'.$$

By a property of Poisson integral, we find

$$\int_{\mathbf{R}^{n-1}} [g(x_1,x',y_1)]^p dx' \leqq M \int_{\mathbf{R}^{n-1}} |f(y)|^p |y_1|^\beta dy'.$$

Applying Lemma 2.1 in Chapter 4, we derive

$$\begin{aligned} &\int_{\mathbf{R}^n} \left(\int_{-\infty}^{\infty} K(x_1,y_1) g(x_1,x',y_1) dy_1 \right)^p dx \\ \leqq\ & \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} K(x_1,y_1) \left(\int_{\mathbf{R}^{n-1}} g(x_1,x',y_1)^p dx' \right)^{1/p} dy_1 \right\}^p dx_1 \\ \leqq\ & M \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} K(x_1,y_1) \left(\int_{\mathbf{R}^{n-1}} |f(y_1,y')|^p |y_1|^\beta dy' \right)^{1/p} dy_1 \right\}^p dx_1 \\ \leqq\ & M {A_K}^p \int_{\mathbf{R}^n} |f(y_1,y')|^p |y_1|^\beta dy' dy_1, \end{aligned}$$

where $A_K = \int_{-\infty}^{\infty} K(1,y_1)|y_1|^{-1/p} dy_1 < \infty$. We thus obtain

$$\begin{aligned} &\left(\int_{\mathbf{R}^n} ||x_1|^{\beta/p} D^\mu (K_{\varepsilon,\ell} * f)(x)|^p dx \right)^{1/p} \\ \leqq\ & \|D^\mu F_{\varepsilon,\ell}\|_p + M^{1/p} A_K \left(\int_{\mathbf{R}^n} |f(y)|^p |y_1|^\beta dy \right)^{1/p} \\ \leqq\ & M \left(\int_{\mathbf{R}^n} |f(y)|^p |y_1|^\beta dy \right)^{1/p}. \end{aligned}$$

Now, letting $\varepsilon \to 0$, we have

$$\left(\int_{\mathbf{R}^n} |D^\mu (U_{\lambda,\ell} f)(x)|^p |x_1|^\beta dx \right)^{1/p} \leqq M \left(\int_{\mathbf{R}^n} |f(y)|^p |y_1|^\beta dy \right)^{1/p}.$$

LEMMA 2.2. *Let* $\xi \in \partial \mathbf{H}$. *If*

$$\int_{\Gamma(\xi,a)\cap B(\xi,1)} |\xi - y|^{m-n} |\nabla^m u(y)| dy < \infty \text{ for every } a > 0,$$

then

$$\int_{\Gamma(\xi,a)\cap B(\xi,1)} |\xi - y|^{k-n} |\nabla^k u(y)| dy < \infty \qquad \text{for every } a > 0 \text{ and } k = 1, ..., m-1.$$

PROOF. It suffices to show that

$$\int_{\Gamma(\xi,a)\cap B(\xi,1)} |\xi - y|^{k-n} |u(y)| dy < \infty$$

whenever $k \geqq 1$, under the assumption that

$$\int_{\Gamma(\xi,a)\cap B(\xi,1)} |\xi - y|^{k+1-n} |\nabla u(y)| dy < \infty.$$

Note that for $|\Theta| = 1$ and $0 < r < 1$,

$$|u(r\Theta) - u(\Theta)| = \left| \int_r^1 (d/dt) u(t\Theta) dt \right| \leqq \int_r^1 |\nabla u(t\Theta)| dt.$$

For simplicity, set $U_m(r) = \int_{\Gamma(\xi,a)\cap \partial B(\xi,1)} |\nabla^m u(r\Theta)| d\Theta$ for $m = 0, 1$. Then we have

$$\begin{aligned} & \int_{\Gamma(\xi,a)\cap B(\xi,1)} |\xi - y|^{k-n} |u(y)| dy = \int_0^1 r^{k-1} U_0(r) dr \\ \leqq \ & U_0(1) \int_0^1 r^{k-1} dr + \int_0^1 r^{k-1} \left(\int_r^1 U_1(t) dt \right) dr \\ = \ & k^{-1} U_0(1) + \int_0^1 U_1(t) \left(\int_0^t r^{k-1} dr \right) dt \\ = \ & k^{-1} U_0(1) + k^{-1} \int_{\Gamma(\xi,a)\cap B(\xi,1)} |\xi - y|^{k+1-n} |\nabla u(y)| dy < \infty. \end{aligned}$$

LEMMA 2.3. *For* $-1 < \beta < p - 1$ *and a nonnegative function* f *satisfying* (2.6), *set*

$$E(f) = \left\{ x \in \partial \mathbf{H} : \int_{B(x,1)} |x - y|^{m-n} f(y) dy = \infty \right\}.$$

Then $C_{m-\beta/p,p}(E(f)) = 0$.

PROOF. Consider

$$u(x) = \int |x-y|^{m-n} f(y)dy$$

for a function f which satisfies (2.6) and vanishes outside a compact set. Then we see from Theorem 2.3 that

$$\int_{\mathbf{R}^n} |\nabla^m u(x)|^p |x_1|^\beta dx < \infty.$$

Set

$$E = \left\{ x : \int_{B(x,1)} |x-y|^{m-\beta/p-n} [|\nabla^m u(y)||y_1|^{\beta/p}] dy = \infty \right\}.$$

Then it follows readily from the definition of $C_{m-\beta/p,p}$ that

$$C_{m-\beta/p,p}(E) = 0.$$

If $x \in \partial\mathbf{H} - E$, then

$$\int_{\Gamma(x,a)} |x-y|^{m-n} |\nabla^m u(y)| dy < \infty.$$

Lemma 2.2 implies that

$$\int_{\Gamma(x,a)} |x-y|^{1-n} |\nabla u(y)| dy < \infty.$$

By a polar coordinate, note that

$$\int_{S(a)} \left(\int_0^1 |\nabla u(x+r\xi)| dr \right) dS(\xi) < \infty$$

with $S(a) = \Gamma(0,a) \cap \mathbf{S}$. This shows that $\lim_{r\to 0} u(x+r\xi)$ exists and is finite for almost every $\xi \in S(a)$, so that it follows that $u(x) < \infty$. Consequently, $E(f) \subseteqq E$ and hence $C_{m-\beta/p,p}(E) = 0$.

We show the existence of perpendicular limits for BLD functions on $\mathbf{H}$, with the aid of integral representations.

THEOREM 2.4. *Let u be a BLD function in $BL_m(L^p_{loc}(\mathbf{H}))$ satisfying* (2.1) *for $0 \leqq \beta < p-1$. Then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $C_{m-\beta/p,p}(E) = 0$ and $\lim_{x_1\to 0} u(x_1, x')$ exists for every $x' \in \mathbf{R}^{n-1}$ such that $(0,x') \notin E$.*

PROOF. As noted above, it suffices to treat

$$U_\lambda f(x) = \int_{B(0,R)} k_\lambda(x,y) f(y) dy$$

for a nonnegative function f satisfying (2.6) and $R > 0$. Write

$$\begin{aligned} U_\lambda f(x) &= \int_{B(x,x_1/2)} k_\lambda(x,y) f(y) dy \\ &\quad + \int_{B(0,R)-B(x,x_1/2)} k_\lambda(x,y) f(y) dy = u_1(x) + u_2(x). \end{aligned}$$

If $\xi = (0, x') \in \partial\mathbf{H} - E(f)$, then u_2 has a nontangential limit $u(\xi)$ at ξ, so that

$$\lim_{t \to 0} u_2(t, x') = u(\xi).$$

For a sequence $\{a_j\}$ of positive numbers, set

$$E_j = \left\{ x : 2^{-j} \leqq x_1 < 2^{-j+1}, u_1(x) > a_j^{-1/p} \right\}.$$

Noting that

$$u_1(x) \leqq \int_{B(0,R)\cap H_j} |x-y|^{m-\beta/p-n} y_1^{\beta/p} f(y) dy$$

with $H_j = \{y : 2^{-j-1} < y_1 < 2^{-j+2}\}$, we have

$$C_{m-\beta/p,p}(E_j; B(0,R)) \leqq a_j \int_{B(0,R)\cap H_j} f(y)^p y_1^\beta dy.$$

By (2.6), take $\{a_j\}$ such that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j \int_{B(0,R)\cap H_j} f(y)^p y_1^\beta dy < \infty.$$

Denote by F^* the projection of a set F to the hyperplane $\partial\mathbf{H}$, and consider

$$E' = \bigcap_{k=1}^{\infty} \left(\bigcup_{j=k}^{\infty} E_j^* \right).$$

Then, in view of Theorem 5.1 in Chapter 5, we obtain

$$C_{m-\beta/p,p}(E'; B(0,R)) \leqq \sum_{j=k}^{\infty} C_{m-\beta/p,p}(E_j^*; B(0,R)) \leqq \sum_{j=k}^{\infty} C_{m-\beta/p,p}(E_j; B(0,R))$$

for any k, which proves

$$C_{m-\beta/p,p}(E'; B(0,R)) = 0.$$

It is easy to see that

$$\lim_{t\to 0} u_1(t, x') = 0$$

for every $(0, x') \in \partial\mathbf{H} - E'$. Thus $E = E(f) \cup E'$ is the required exceptional set.

REMARK 2.1. Suppose $E \subsetneqq \partial\mathbf{H}$ and $C_{m,p}(E) = 0$. Then $C_{m,p}(E_j) = 0$, where $E_j = \{\xi + 2^{-j}e : \xi \in E\}$. Hence we can find a nonnegative function f_j such that $f_j = 0$ outside H_j, $U_m f_j = \infty$ on E_j and

$$\int_{H_j} f(y)^p dy < 2^{j\beta} 2^{-j}.$$

Then, setting $f = \sup_j f_j$, we see that $U_m f(x) = \infty$ for all $x \in \bigcup_j E_j$ and

$$\int f(y)^p |y_1|^\beta dy < \infty.$$

If $-1 < \beta < 0$, then $U_m f$ satisfies (2.1) by Theorem 2.3 and moreover

$$\limsup_{t \to 0} U_m f(\xi + te) = \infty \qquad \text{for all } \xi \in E.$$

With the aid of Corollary 2.1, we can prove the following result.

COROLLARY 2.2. *Let u be a BLD function in $BL_m(L^p_{loc}(\mathbf{H}))$ satisfying (2.1) for $-1 \leqq \beta < mp-1$. Let ℓ be the nonnegative integer such that $\ell p - 1 \leqq \beta < (\ell+1)p-1$.*

(i) *If $\ell p \leqq \beta < (\ell+1)p-1$, then there exists a set $E \subsetneqq \partial\mathbf{H}$ such that $C_{m-\beta/p,p}(E) = 0$ and $\lim_{x_1 \to 0} u(x_1, x')$ exists for every $x' \in \mathbf{R}^{n-1}$ such that $(0, x') \notin E$.*

(ii) *If $\beta < \ell p$, then there exists a set $E \subsetneqq \partial\mathbf{H}$ such that $C_{m-\ell,p}(E) = 0$ and $\lim_{x_1 \to 0} u(x_1, x')$ exists for every $x' \in \mathbf{R}^{n-1}$ such that $(0, x') \notin E$.*

We next study the existence of limits along curves tangential to the hyperplane $\partial\mathbf{H}$. Let ψ_1 be a positive nondecreasing function on $(0, \infty)$ satisfying the doubling condition :

$$\psi_1(2r) \leqq M\psi_1(r) \qquad \text{for } r > 0.$$

Assume further that $r^{-1}\psi_1(r)$ is nondecreasing on $(0, \infty)$. Let ψ_j, $j = 2, ..., n-1$, be functions on $[0, \infty)$ such that $\psi_j(0) = 0$ and

$$|\psi_j(s) - \psi_j(t)| \leqq M|s - t| \qquad \text{whenever } 0 \leqq s < t < \infty.$$

Setting $\psi_n(r) = r$, we define

$$\Psi(r) = (\psi_1(r), ..., \psi_n(r))$$

and

$$\xi(r) = \xi + \Psi(r)$$

for $\xi \in \partial\mathbf{H}$.

THEOREM 2.5. *Let $0 < \beta < p-1$, $n - mp + \beta \geqq 0$ and $h(r) = [\psi_1(r)]^{n-mp+\beta}$. If u is a BLD function in $BL_m(L^p_{loc}(\mathbf{H}))$ satisfying (2.1), then there exist sets E_1, $E_2 \subsetneqq \partial\mathbf{H}$ such that $C_{m-\beta/p,p}(E_1) = 0$, $H_h(E_2) = 0$ and*

$$\lim_{r \to 0} u(\xi(r)) \quad \text{exists for every } \xi \in \partial\mathbf{H} - (E_1 \cup E_2).$$

REMARK 2.2. If $\psi_1(r) = r^\gamma$ with $\gamma > 1$ and $n - mp + \beta > 0$, then $H_h(E_1) = 0$ by Theorem 2.3 in Chapter 5; if $\psi_1(r) = r$, then $C_{\alpha-\beta/p,p}(E_2) = 0$ by Theorem 2.2 in Chapter 5.

PROOF OF THEOREM 2.5. We are only concerned with the function

$$U_\lambda f(x) = \int k_\lambda(x,y) f(y) dy$$

for a nonnegative function f satisfying (2.6) and vanishing outside $B(0,R)$. For $a = 10^{-1}$, write

$$\begin{aligned} U_\lambda f(x) &= \int_{B(x,ax_1)} k_\lambda(x,y) f(y) dy \\ &\quad + \int_{B(\xi,2|\xi-x|)-B(x,ax_1)} k_\lambda(x,y) f(y) dy \\ &\quad + \int_{B(0,R)-B(\xi,2|\xi-x|)} k_\lambda(x,y) f(y) dy = u_1(x) + u_2(x) + u_3(x). \end{aligned}$$

If $\xi \in \partial\mathbf{H} - E(f)$, then u_3 has a limit $u(\xi)$ at ξ by Lebesgue's dominated convergence theorem, so that

$$\lim_{r\to 0} u_3(\xi(r)) = u(\xi).$$

By Hölder's inequality we have

$$\begin{aligned} |u_2(x)| &\leqq \int_{B(\xi,2|\xi-x|)-B(x,ax_1)} |x-y|^{m-n} |f(y)| dy \\ &\leqq \left(\int_{B(\xi,2|\xi-x|)-B(x,ax_1)} |x-y|^{p'(m-n)} |y_1|^{-p'\beta/p} dy \right)^{1/p'} \\ &\quad \times \left(\int_{B(\xi,2|\xi-x|)-B(x,ax_1)} |f(y)|^p |y_1|^\beta dy \right)^{1/p} \\ &\leqq M {x_1}^{m-(n+\beta)/p} \left(\int_{B(\xi,2|\xi-x|)} |f(y)|^p |y_1|^\beta dy \right)^{1/p}. \end{aligned}$$

Define

$$E' = \left\{ \xi \in \partial\mathbf{H} : \limsup_{r\to 0} h(r)^{-1} \int_{B(\xi,r)} |f(y)|^p |y_1|^\beta dy > 0 \right\}.$$

Then $H_h(E') = 0$. Further, if $\xi \in \partial\mathbf{H} - E'$, then

$$\lim_{r\to 0} u_2(\xi(r)) = 0.$$

By (2.6), take a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j \int_{H_j} f(y)^p y_1^\beta dy < \infty.$$

As in the previous proof, consider the sets

$$X_j = \{x : 2^{-j} \leqq x_1 < 2^{-j+1}, u_1(x) > a_j^{-1/p}\}.$$

If $x \in X_j$, then

$$\begin{aligned} a_j^{-1/p} &< \int_{B(x,ax_1)} |x-y|^{m-n}|f(y)|dy \\ &= (ax_1)^{m-n}F(x,ax_1) + \int_0^{ax_1} F(x,r)d(-r^{m-n}), \end{aligned}$$

where $F(x,r) = \int_{B(x,r)} |f(y)|dy$. By Hölder's inequality we have

$$F(x,r) \leqq |B(x,r)|^{1/p'} \left(\int_{B(x,r)} |f(y)|^p dy \right)^{1/p},$$

so that we can find $r = r(x)$ such that $0 < r \leqq ax_1$ and

$$\int_{B(x,r)} |f(y)|^p dy \geqq M a_j^{-1} x_1^{-\beta} r^{n-mp+\beta}.$$

Since $\{B(x, r(x)) : x \in X_j\}$ covers X_j, by a covering lemma (see Theorem 10.1 in Chapter 1), we can choose a disjoint family $\{B_{j,\ell}\}$, $B_{j,\ell} = B(x_{j,\ell}, r(x_{j,\ell}))$, for which $\{5B_{j,\ell}\}$ covers X_j. For $x \in \mathbf{H}$, let $\tilde{x}$ be the point on $\partial\mathbf{H}$ such that $x = \tilde{x}(r)$ for some $r > 0$, and denote by $\tilde{X}$ the set of all $\tilde{x}$ for $x \in X$. Now consider the set

$$E'' = \bigcap_{k=1}^{\infty} \left(\bigcup_{j=k}^{\infty} \tilde{X}_j \right).$$

Find $t_{j,\ell}$ with $\psi_1(t_{j,\ell}) = r(x_{j,\ell})$. Then $r(x_{j,\ell}) \leqq [1/\psi_1(1)]t_{j,\ell}$. If $y = \zeta(s) \in B(x_{j,\ell}, r(x_{j,\ell}))$ with $\zeta \in \partial\mathbf{H}$ and $x_{j,\ell} = \tilde{x}_{j,\ell}(s_{j,\ell})$, then

$$\psi_1(|s_{j,\ell} - s|) \leqq |\psi_1(s_{j,\ell}) - \psi_1(s)| \leqq |x_{j,\ell} - y| < r(x_{j,\ell}) = \psi_1(t_{j,\ell}),$$

so that

$$|\tilde{x}_{j,\ell} - \zeta| \leqq |x_{j,\ell} - y| + \sum_{i=2}^{n} |\psi_j(s_{j,\ell}) - \psi_i(s)| \leqq r(x_{j,\ell}) + M|s_{j,\ell} - s| \leqq M t_{j,\ell}.$$

This implies that $B(\tilde{x}_{j,\ell}, Mt_{j,\ell}) \supseteqq \tilde{B}_{j,\ell}$. On the other hand we have

$$\begin{aligned} \sum_{\ell} [5r(x_{j,\ell})]^{n-mp+\beta} &\leqq M a_j 2^{-j\beta} \sum_{\ell} \int_{B_{j,\ell}} |f(y)|^p dy \\ &\leqq M a_j \int_{H_j} |f(y)|^p y_1^{\beta} dy. \end{aligned}$$

Since $h(Mt_{j,\ell}) \leqq Mr(x_{j,\ell})^{n-mp+\beta}$, we find

$$H_h^{(\delta_k)}(E'') \leqq M \sum_{j=k}^{\infty} a_j \int_{H_j} |f(y)|^p y_1^\beta dy,$$

where $\delta_k = \sup\{Mt_{j,\ell} : j \geqq k\}$. Thus it follows that

$$H_h(E'') = 0.$$

If $\xi \in \partial\mathbf{H} - E''$, then

$$\lim_{r\to 0} u_1(\xi(r)) = 0.$$

Now $E_1 = E(f)$ and $E_2 = E' \cup E''$ have all the required properties.

By Lemma 1.4, we have

LEMMA 2.4. *Let $n - mp + \beta \geqq 0$. For a nonnegative function f satisfying* (2.6), *set*

$$F_{\beta'} = \left\{ \xi \in \partial\mathbf{H} : \int_{B(\xi,1)} |\xi - y|^{mp-\beta'-n} f(y)^p |y_1|^{\beta'} dy = \infty \right\}.$$

If $\beta' > \beta$, then $H_{n-mp+\beta}(F_{\beta'}) = 0$.

THEOREM 2.6. *Let $-1 < \beta < p-1$ and $n - mp + \beta \geqq 0$. If u is a BLD function in $BL_m(L^p_{loc}(\mathbf{H}))$ satisfying* (2.1), *then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $C_{m-\beta/p,p}(E) = 0$ and for any $\xi \in \partial\mathbf{H} - E$, $\lim_{r\to 0} u(\xi + r\zeta)$ exists for every $\zeta \in \mathbf{H} \cap \mathbf{S} - E(\xi)$, where $C_{m,p}(E(\xi)) = 0$.*

The proof is similar to that of Theorem 1.5, by Lemma 2.4 together with Theorem 6.1 in Chapter 5, because $C_{m-\beta/p,p}(F_{\beta'}) = 0$ and if $\xi \in \partial\mathbf{H} - F_{\beta'}$, then

$$\int_{\Gamma(\xi,a)\cap B(\xi,1)} |\xi - y|^{mp-n} f(y)^p dy < \infty \qquad \text{for any } a > 0,$$

which plays the same role as (4.4) in Chapter 5.

COROLLARY 2.3. *Let u be a BLD function in $BL_m(L^p_{loc}(\mathbf{H}))$ satisfying* (2.1) *for $-1 \leqq \beta < mp-1$. Let ℓ be the nonnegative integer such that $\ell p - 1 \leqq \beta < (\ell+1)p - 1$.*

(i) *If $\ell p - 1 < \beta$, then there exists a set $E \subseteqq \mathbf{H}$ such that $C_{m-\beta/p,p}(E) = 0$ and for every $\xi \in \mathbf{H} - E$, $\lim_{r\to 0} u(\xi + r\zeta)$ exists for every $\zeta \in \mathbf{H} \cap \mathbf{S} - E(\xi)$, where $C_{m,p}(E(\xi)) = 0$.*

(ii) *If $\beta = \ell p - 1$, then there exists a set $E \subseteqq \mathbf{H}$ such that E has Hausdorff dimension at most $n - mp + \beta$ and for every $\xi \in \mathbf{H} - E$, $\lim_{r\to 0} u(\xi + r\zeta)$ exists for every $\zeta \in \mathbf{H} \cap \mathbf{S} - E(\xi)$, where $C_{m,p}(E(\xi)) = 0$.*

Finally we discuss the existence of L^q-mean limits of BLD functions at the boundary points. Let us say that a function u has an L^q-mean limit ℓ if

$$\lim_{r\to 0} \frac{1}{|B_+(\xi,r)|}\int_{B_+(\xi,r)} |u(x)-\ell|^q dx = 0,$$

where $B_+(\xi,r) = \mathbf{H}\cap B(\xi,r)$ as before.

LEMMA 2.5. *Let* $0 \leqq \gamma < 1$, $\gamma/p' < \alpha < n$ *and*

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}\left(\alpha - \frac{\gamma}{p'}\right).$$

For a nonnegative function $f \in L^p(\mathbf{R}^n)$, *set*

$$F(x) = \int |x-y|^{\alpha-n} f(y)|y_1|^{-\gamma/p'} dy.$$

Then

$$\|F\|_q \leqq M\|f\|_p.$$

In case $\gamma = 0$, this is just Sobolev's inequality. For the general case, the proof is left to the reader.

Define

$$\frac{1}{p^*} = \frac{1}{p} - \frac{m}{n}$$

and

$$\frac{1}{p^{**}} = \frac{1}{p} - \frac{1}{n}\left(m - \frac{\beta}{p}\right).$$

THEOREM 2.7. *Let* $-1 < \beta < p-1$. *If* u *is a BLD function in* $BL_m(L^p_{loc}(\mathbf{H}))$ *satisfying* (2.1), *then there exists a set* $E \subsetneqq \partial\mathbf{H}$ *such that* $C_{m-\beta/p,p}(E) = 0$ *and* u *has an* L^q*-mean limit at* $\xi \in \partial\mathbf{H} - E$, *where* q *is given as follows :*

- (i) $q = p^{**}$ *when* $\beta \geqq 0$ *and* $mp - \beta < n$;
- (ii) $1 < q < \infty$ *when* $\beta \geqq 0$ *and* $mp - \beta = n$;
- (iii) $q = \infty$ *when* $\beta \geqq 0$ *and* $mp - \beta > n$;
- (iv) $q = p^*$ *when* $\beta < 0$ *and* $mp < n$;
- (v) $1 < q < \infty$ *when* $\beta < 0$ *and* $mp = n$;
- (vi) $q = \infty$ *when* $\beta < 0$ *and* $mp > n$.

REMARK 2.3. In case $mp - \beta > n$, if $C_{m-\beta/p,p}(E) = 0$, then E is empty and, in fact, we show soon that u is continuous in case (iii) and (vi).

PROOF OF THEOREM 2.7. We may suppose that u is of the form

$$u(x) = U_\lambda f(x) = \int k_\lambda(x,y) f(y)|y_1|^{-\beta/p} dy$$

for $|\lambda| = m$ and a nonnegative function $f \in L^p(\mathbf{R}^n)$ vanishing outside a compact set. Write

$$\begin{aligned} u(x) &= \int_{B(\xi,2|\xi-x|)} k_\lambda(x,y)f(y)|y_1|^{-\beta/p}dy \\ &\quad + \int_{\mathbf{R}^n - B(\xi,2|\xi-x|)} k_\lambda(x,y)f(y)|y_1|^{-\beta/p}dy = u_1(x) + u_2(x). \end{aligned}$$

Define

$$E_1 = \left\{ \xi \in \partial\mathbf{H} : \int_{B(\xi,1)} |\xi - y|^{m-n}[f(y)|y_1|^{-\beta/p}]dy = \infty \right\}$$

and

$$E_2 = \left\{ \xi \in \partial\mathbf{H} : \limsup_{r\to 0} \ r^{mp-n-\beta} \int_{B(\xi,r)} |f(y)|^p dy > 0 \right\}.$$

Then $C_{m-\beta/p,p}(E_1) = 0$ by Lemma 2.3. On the other hand, $H_{n-mp+\beta}(E_2) = 0$, which proves $C_{m-\beta/p,p}(E_2) = 0$ with the aid of Theorem 2.2 in Chapter 5. If $\xi \in \partial\mathbf{H} - E_1$, then u_2 has a limit $u(\xi)$ at ξ by Lebesgue's dominated convergence theorem, so that

$$\lim_{r\to 0} \frac{1}{|B_+(\xi,r)|} \int_{B_+(\xi,r)} |u_2(y) - u(\xi)|^q dx = 0$$

for any $q > 1$. Thus it suffices to show that

$$\lim_{r\to 0} \frac{1}{|B_+(\xi,r)|} \int_{B_+(\xi,r)} |u_1|^q dx = 0$$

whenever $\xi \in \partial\mathbf{H} - E_2$, for q described as in the theorem.

Case 1 : $\beta \geqq 0$ and $mp - \beta < n$. In this case we have by Lemma 2.5

$$\begin{aligned} \frac{1}{|B_+(\xi,r)|} \int_{B_+(\xi,r)} |u_1|^{p^{**}} dx &\leqq Mr^{-n} \int \left(\int_{B(\xi,2r)} |x-y|^{m-n}|f(y)||y_1|^{-\beta/p}dy \right)^{p^{**}} dx \\ &\leqq M \left(r^{mp-\beta-n} \int_{B(\xi,2r)} |f(y)|^p dy \right)^{p^{**}/p} \to 0 \quad \text{as } r \to 0. \end{aligned}$$

Case 2 : $\beta \geqq 0$ and $mp - \beta = n$. For any q, $1 < q < \infty$, let $\varepsilon = pnq^{-1}$. If q is so large that $\beta + \varepsilon < p - 1$, then we have by Lemma 2.5

$$\begin{aligned} \frac{1}{|B_+(\xi,r)|} \int_{B_+(\xi,r)} |u_1|^{p^*} dx &\leqq Mr^{-n} \left(\int_{B(\xi,2r)} |f(y)|^p |y_1|^{\varepsilon} dy \right)^{q/p} \\ &\leqq M \left(\int_{B(\xi,2r)} |f(y)|^p dy \right)^{q/p} \to 0 \qquad \text{as } r \to 0. \end{aligned}$$

Case 3 : $\beta < 0$ and $mp < n$. As in Case 1, we have by Lemma 2.5

$$\begin{aligned} \frac{1}{|B_+(\xi,r)|} \int_{B_+(\xi,r)} |u_1|^{p^*} dx &\leqq Mr^{-n} \left(\int_{B(\xi,2r)} |f(y)|^p |y_1|^{-\beta} dy \right)^{p^*/p} \\ &\leqq M \left(r^{mp-\beta-n} \int_{B(\xi,2r)} |f(y)|^p dy \right)^{p^*/p} \to 0 \end{aligned}$$

as $r \to 0$.

Case 4 : $\beta < 0$ and $mp = n$. For $p < q < \infty$, let $1/q = 1/p - \varepsilon/n$ with $\varepsilon > 0$. Then we have by Lemma 2.5

$$\begin{aligned} \frac{1}{|B_+(\xi, r)|} \int_{B_+(\xi,r)} |u_1|^q dx &\leqq Mr^{-n} \int \left(r^{m-\varepsilon} \int_{B(\xi,2r)} |x-y|^{\varepsilon-n} |f(y)||y_1|^{-\beta/p} dy \right)^q dx \\ &\leqq M \left(r^{mp-\beta-n} \int_{B(\xi,2r)} |f(y)|^p dy \right)^{q/p} \to 0 \qquad \text{as } r \to 0. \end{aligned}$$

In case $mp > n$ and $mp - \beta > n$, we have

$$\begin{aligned} & \int_{B(x,r)} |x-y|^{m-n} |f(y)||y_1|^{-\beta/p} dy \\ \leqq \; & M \left(\int_{B(x,r)} |x-y|^{p'(m-n)} |y_1|^{-p'\beta/p} dy \right)^{1/p'} \left(\int_{B(\xi,2r)} |f(y)|^p dy \right)^{1/p} \\ \leqq \; & M r^{(mp-\beta-n)/p} \|f\|_p \end{aligned}$$

whenever $0 < |x_1| < r$. This implies that u is continuous on $\mathbf{R}^n$.

We discuss the existence of L^q-mean differentiability of BLD functions at the boundary points. We say that a function u is ℓ times L^q-mean differentiable at ξ if

$$\lim_{r \to 0} r^{-n-\ell} \int_{B_+(\xi,r)} |u(x) - P_\ell(x)|^q dx = 0$$

for some polynomial P_ℓ of degree at most ℓ.

In view of the proofs of Theorem 2.7 and Theorem 7.1 in Chapter 5, we can prove the following result.

THEOREM 2.8. *Let $-1 < \beta < p - 1$. If u is a BLD function in $BL_m(L^p_{loc}(\mathbf{H}))$ satisfying* (2.1)*, then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $C_{m-\ell-\beta/p,p}(E) = 0$ and u is ℓ times L^q-mean differentiable at $\xi \in \partial\mathbf{H} - E$, where q is given as follows :*

(i) $q = p^{**}$ *when* $\beta \geqq 0$ *and* $mp - \beta < n$;
(ii) $1 < q < \infty$ *when* $\beta \geqq 0$ *and* $mp - \beta = n$;
(iii) $q = \infty$ *when* $\beta \geqq 0$ *and* $mp - \beta > n$;
(iv) $q = p^*$ *when* $\beta < 0$ *and* $mp < n$;
(v) $1 < q < \infty$ *when* $\beta < 0$ *and* $mp = n$;
(vi) $q = \infty$ *when* $\beta < 0$ *and* $mp > n$.

REMARK 2.4. In case (iii) and (vi), u is shown to be ℓ times differentiable at $\xi \in \partial\mathbf{H} - E$.

8.3 Nontangential and tangential limits

If $u \in BL_m(L^p_{loc}(\mathbf{H}))$ and $mp > n$, then u is equal to a continuous function almost everywhere on $\mathbf{H}$, and hence it may be expected that u has better boundary limits than those in the previous section.

To obtain general results, we consider the function of the form

$$U_k f(x) = \int k(x-y) f(y) dy,$$

where $k(x)$ is a function on $\mathbf{R}^n$ which is continuous on $\mathbf{R}^n$ except at the origin and

$$(3.1) \qquad |k(x)| \leqq M|x|^{\alpha-n}, \qquad 0 < \alpha < n.$$

In this section, assume that

$$(3.2) \qquad \int_{\mathbf{R}^n} \Phi_p(|f(y)|)|y_1|^\beta dy < \infty$$

and

$$(3.3) \qquad \int_0^1 [t^{n-\alpha p}\varphi(t^{-1})]^{-1/(p-1)} t^{-1} dt < \infty,$$

where φ is a function as in Chapter 5 and $\Phi_p(r) = r^p\varphi(r)$. Then it is seen (Theorem 3.1 in Chapter 5) that the potential $U_k f$ is continuous outside the hyperplane $\partial\mathbf{H}$. Define the capacity

$$C_{\alpha,\Phi_p,\beta}(E;G) = \inf \int_{\mathbf{R}^n} \Phi_p(g(y))|y_1|^\beta dy,$$

where the infimum is taken over all nonnegative measurable functions g for which $U_\alpha g \geqq 1$ on E and $g = 0$ outside G. As before, we write $C_{\alpha,\Phi_p,\beta}(E) = 0$ if $C_{\alpha,\Phi_p,\beta}(E\cap G;G) = 0$ for all bounded open set G.

The following can be proved in the same way as Theorem 2.1 in Chapter 5.

LEMMA 3.1. *Suppose* $-1 < \beta < p-1$ *and* $n - \alpha p + \beta \geqq 0$. *Set*

$$h(r) = h(r;R) = \left(\int_r^R [t^{n-\alpha p+\beta}\varphi(t^{-1})]^{-1/(p-1)} t^{-1} dt \right)^{1-p}$$

for $0 < r \leqq R/2$. *If* $B(x,r) \subseteqq B(0,R/2)$, *then*

$$M^{-1}h(r) \leqq C_{\alpha,\Phi_p,\beta}(B(x,r);B(0,R)) \leqq Mh(r).$$

In view of Theorem 2.2 in Chapter 5, we have the following.

COROLLARY 3.1. *If* $H_h(E) < \infty$, *then* $C_{\alpha,\Phi_p,\omega}(E) = 0$.

LEMMA 3.2. *Let $-1 < \beta < p-1$ and $n-\alpha p+\beta \geqq 0$. If $E \subseteqq \partial\mathbf{H}$ and $C_{\alpha,\Phi_p,\beta}(E) = 0$, then E has Hausdorff dimension at most $n - \alpha p + \beta$.*

PROOF. Let $\varepsilon > 0$, $\beta < \gamma < p-1$ and $\theta = [n(p-1) - (\gamma+\varepsilon)]/p(n-\alpha)$. Since $C_{\alpha,\Phi_p,\beta}(E) = 0$, we can find a nonnegative measurable function f satisfying (3.2) such that $U_\alpha f \not\equiv \infty$ but $U_\alpha f = \infty$ on E. Then we have by Hölder's inequality

$$\begin{aligned} \int_{B(x,1)} |x-y|^{\alpha-n} f(y)dy &\leqq \left(\int_{B(x,1)} [|x-y|^{\theta(\alpha-n)} |y_1|^{-\gamma/p}]^{p'} dy \right)^{1/p'} \\ &\times \left(\int_{B(x,1)} |x-y|^{p(1-\theta)(\alpha-n)} f(y)^p |y_1|^\gamma dy \right)^{1/p} \\ \leqq\ & M \left(\int_{B(x,1)} |x-y|^{-(n-\alpha p+\gamma+\varepsilon)} f(y)^p |y_1|^\gamma dy \right)^{1/p}, \end{aligned}$$

since $p'[\theta(\alpha-n) - \gamma/p] + n = p'\varepsilon/p > 0$. Thus, in view of Lemma 1.4, we see that $H_{n-\alpha p+\beta+\varepsilon}(E) = 0$, which implies that E has Hausdorff dimension at most $n-\alpha p+\beta$.

In case α is a positive integer m, we can prove the following result.

LEMMA 3.3. *Suppose $-1 < \beta < p-1$. If $E \subseteqq \partial\mathbf{H}$ and $C_{m,\Phi_p,\beta}(E) = 0$, then $C_{m-\beta/p,\Phi_p}(E) = 0$.*

PROOF. If $C_{m,\Phi_p,\beta}(E) = 0$, then we can find a nonnegative measurable function f satisfying (3.2) such that $U_m f \not\equiv \infty$ but $U_m f = \infty$ on E. We may assume further that f has compact support and then

$$\int \Phi_p(f(y)|y_1|^{\beta/p})dy < \infty.$$

For $\gamma = \beta/p$, we have by Theorem 2.3

$$\int [|\nabla^m U_m f(x)||x_1|^\gamma]^q dx \leqq M \int [f(y)||y_1|^\gamma]^q dy$$

whenever $q > 1$ and $-1 < \gamma q < q-1$. Now let $1 < q_1 < p < q_2$, $-1/q_2 < \gamma < 1/q_1'$,

$$f_{1,a}(y) = \begin{cases} f(y) & \text{when } f(y)|y_1|^\gamma \geqq a, \\ 0 & \text{otherwise,} \end{cases}$$

and $f_{2,a} = f - f_{1,a}$ for $a > 0$. Then

$$\begin{aligned} \int \Phi_p(|\nabla_m U_m f(x)||x_1|^\gamma)dx &= \int |\{x : |\nabla_m U_m f(x)||x_1|^\gamma > a\}| d\Phi_p(a) \\ &\leqq M \int [f(y)|y_1|^\gamma]^{q_1} \left(\int_0^{f(y)|y_1|^\gamma} a^{-q_1} d\Phi_p(a) \right) dy \end{aligned}$$

$$+M \int [f(y)|y_1|^\gamma]^{q_2} \left(\int_{f(y)|y_1|^\gamma}^\infty a^{-q_2} d\Phi_p(a) \right) dy$$
$$\leqq M \int \Phi_p(f(y)|y_1|^\gamma) dy < \infty.$$

This implies that

$$\int_G \Phi_p(|\nabla_m U_m f(x)|)|x_1|^{\gamma p} dx < \infty$$

for every bounded open set G in $\mathbf{R}^n$. As in the proof of Lemma 2.3, we see that E is included in

$$E' = \left\{ x : \int_{B(x,1)} |x-y|^{m-\beta/p-n} [|\nabla^m U_m f(y)||y_1|^\gamma] dy = \infty \right\},$$

which satisfies $C_{m-\beta/p,\Phi_p}(E') = 0$.

We also define a nondecreasing function τ by setting

$$\tau(r) = \inf_{r<s<1} s^\beta \int_0^s [t^{n-\alpha p} \varphi(t^{-1})]^{-1/(p-1)} t^{-1} dt$$

for $0 < s \leqq 1/2$; set $\tau(r) = \tau(1/2)$ for $r > 1/2$.

Let ψ be a continuous nondecreasing function on $[0, \infty)$ such that $\psi(r) > 0$ and $r^{-1}\psi(r)$ is nondecreasing for $r > 0$. For $\xi \in \partial \mathbf{H}$, set

$$T_\psi(\xi, a) = \{x \in \mathbf{H} : \psi(|x-\xi|) < a x_1\}.$$

In case $\psi(r) \equiv r$, the set is nothing but a cone with vertex at ξ. We say that u has a T_ψ-limit ℓ at ξ if

$$\lim_{x \to \xi, x \in T_\psi(\xi,a)} u(x) = \ell \qquad \text{for all } a > 0;$$

in case $\psi(r) = r^\gamma$ for $\gamma \geqq 1$, T_ψ-limit is called T_γ-limit. Note that T_1-limits are nothing but nontangential limits. We also say that u has a T_∞-limit at ξ if it has a T_γ-limit at ξ for any $\gamma > 1$.

Theorem 3.1. *Let f satisfy (3.2) for $-1 < \beta < p-1$ and $U_\alpha|f| \not\equiv \infty$. Set $h_\gamma(r) = \tau(r^\gamma)$ for $\gamma \geqq 1$.*

(i) *If $n-\alpha p+\beta > 0$ and $\gamma > 1$, then there exists a set $E \subseteqq \partial \mathbf{H}$ such that $H_{h_\gamma}(E) = 0$ and $U_k f$ has a T_γ-limit at any $\xi \in \partial \mathbf{H} - E$.*

(ii) *If $n-\alpha p+\beta > 0$, then there exists a set $E \subseteqq \partial \mathbf{H}$ such that $C_{\alpha,\Phi_p,\beta}(E) = 0$ and $U_k f$ has a nontangential limit at any $\xi \in \partial \mathbf{H} - E$.*

(iii) *If $n-\alpha p+\beta = 0$, then there exists a set $E \subseteqq \partial \mathbf{H}$ such that $C_{\alpha,\Phi_p,\beta}(E) = 0$ and $U_k f$ has a T_∞-limit at any $\xi \in \partial \mathbf{H} - E$.*

(iv) *If $n-\alpha p+\beta < 0$ or $\beta = \alpha p - n = 0$, then $U_k f$ has a finite limit at any $\xi \in \partial \mathbf{H}$.*

PROOF. Consider the sets

$$E_1 = \left\{ x : \int_{B(x,1)} |x-y|^{\alpha-n} |f(y)| dy = \infty \right\}$$

and

$$E_2 = \left\{ x : \limsup_{r \to 0} \, [h(r)]^{-1} \int_{B(x,r)} \Phi_p(|f(y)|) |y_1|^\beta dy > 0 \right\}$$

with h as in Lemma 3.1. Here note that $C_{\alpha,\Phi_p,\beta}(E_1) = 0$ and $H_h(E_2) = 0$. With the aid of Lemmas 3.1 and 3.2, it suffices to show that $U_k f$ has a T_ψ-limit at every $\xi \in \partial \mathbf{H} - (E_1 \cup E_2)$. As before, write

$$\begin{aligned} U_k f(x) &= \int_{B(\xi, 2|x-\xi|)} k(x-y) f(y) dy \\ &\quad + \int_{\mathbf{R}^n - B(\xi,2|x-\xi|)} k(x-y) f(y) dy = u_1(x) + u_2(x). \end{aligned}$$

Then, in view of Lebesgue's dominated convergence theorem, we see that u_2 has a finite limit at $\xi \in \partial \mathbf{H} - E_1$. On the other hand, the proof of Theorem 3.1 in Chapter 5 shows that

$$|u_2(x)| \leqq M \left([\tau(x_1)]^{-1} \int_{B(\xi,2|x-\xi|)} \Phi_p(|f(y)|) |y_1|^\beta dy \right)^{1/p} + M |x-\xi|^{\alpha-\delta}$$

for $0 < \delta < \alpha$. Hence u_1 has limit zero at $\xi \in \partial \mathbf{H} - E_2$.

THEOREM 3.2. *Let $-1 \leqq \beta < p-1$, and u be a BLD function on $\mathbf{H}$ satisfying*

$$\int_G \Phi_p(|\nabla_m u(x)|) |x_1|^\beta dx < \infty \text{ for any bounded open set } G \subseteqq \mathbf{H}. \tag{3.4}$$

Let ℓ be the nonnegative integer such that $\ell p - 1 \leqq \beta < (\ell+1)p - 1$, and $h_\gamma(r) = r^{\gamma(n-mp+\beta)}$.

(i) *If $n - mp + \beta > 0$, $\ell p - 1 < \beta$ and $\gamma > 1$, then there exists a set $E \subseteqq \partial \mathbf{H}$ such that $H_{h_\gamma}(E) = 0$ and u has a T_γ-limit at any $\xi \in \partial \mathbf{H} - E$.*

(ii) *If $n - mp + \beta > 0$ and $\ell p - 1 < \beta$, then there exists a set $E \subseteqq \partial \mathbf{H}$ such that $C_{m-\beta/p,\Phi_p}(E) = 0$ and u has a nontangential limit at any $\xi \in \partial \mathbf{H} - E$.*

(iii) *If $n - mp + \beta \geqq 0$ and $\beta = \ell p - 1$, then there exists a set $E \subseteqq \partial \mathbf{H}$ such that E has Hausdorff dimension at most $\gamma(n - mp + \beta)$ and u has a T_γ-limit at any $\xi \in \partial \mathbf{H} - E$.*

(iv) *If $mp - n = \beta > \ell p - 1$, then there exists a set $E \subseteqq \partial \mathbf{H}$ such that $C_{m-\beta/p,\Phi_p}(E) = 0$ and u has a T_∞-limit at any $\xi \in \partial \mathbf{H} - E$.*

(iv) *If $n - mp + \beta < 0$ or $\beta = mp - n = 0$, then u has a finite limit at any $\xi \in \partial \mathbf{H}$.*

For Green potentials, we have the following.

THEOREM 3.3. *Let f satisfy (3.2) for $-1 < \beta \leqq 2p-1$ and $h(r) = \tau(\psi(r))$. Suppose further $G_\alpha f \not\equiv \infty$ on $\mathbf{H}$.*

(i) *If $n - \alpha p + \beta > 0$, then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $H_h(E) = 0$ and $G_\alpha f$ has T_ψ-limit zero at any $\xi \in \partial\mathbf{H} - E$.*

(ii) *If $n - \alpha p + \beta \leqq 0$, then $G_\alpha f$ has limit zero at any $\xi \in \partial\mathbf{H}$.*

8.4 Polyharmonic functions

In this chapter we are concerned with polyharmonic functions on the half space $\mathbf{H}$. First we prepare the fundamental properties for polyharmonic functions; the main purpose is to give mean-value inequality. Recall that an infinitely differentiable function u on an open set G is called polyharmonic of order m in G if

$$\Delta^m u = 0 \qquad \text{on } G.$$

LEMMA 4.1. *If u is polyharmonic of order m in G, then $|x - x_0|^{2k} u(x)$ is polyharmonic of order $m + k$ on G for any fixed x_0.*

PROOF. If $k = 1$, then

$$\Delta(|x - x_0|^2 u(x)) = 2nu(x) + 4\sum_{j=1}^{n} (x_j - x_{0j}) D_j u(x),$$

where $D_j = (\partial/\partial x_j)$. Note that

$$\Delta^m \left(\sum_{j=1}^{n} (x_j - x_{0j}) D_j u(x) \right) = 2m\Delta^m u + \sum_{j=1}^{n} (x_j - x_{0j}) \Delta^m (D_j u)(x).$$

Hence it follows that $|x - x_0|^2 u$ is polyharmonic of order $m + 1$ on G whenever u is polyharmonic of order m on G, and the case $k = 1$ is shown. Now apply induction on k. In fact, assume that $v(x) = |x - x_0|^{2k} u(x)$ is polyharmonic of order $m + k$; then the above considerations prove that $|x - x_0|^{2(k+1)} u(x) = |x - x_0|^2 v(x)$ is polyharmonic of order $(m + k) + 1$.

LEMMA 4.2 (finite Almansi expansion). *Let G be a star domain with center at 0, that is, $tx \in G$ whenever $x \in G$ and $0 \leqq t \leqq 1$. If u is polyharmonic of order m in G, then there exist harmonic functions $h_1, h_2, \ldots, h_m$ such that*

$$u(x) = \sum_{k=1}^{m} |x|^{2k-2} h_k(x) \qquad \text{for } x \in G.$$

PROOF. We show this lemma by induction on m. First note that the case $m=1$ is trivially true. Hence assume that the assertion is true for m, and let u be polyharmonic of order $m+1$ in G. Since Δu is polyharmonic of order m, by assumption on induction we can find harmonic functions $h_1,\dots,h_m$ for which

$$\Delta u(x) = \sum_{k=1}^{m} |x|^{2k-2} h_k(x).$$

If $u(x) = \sum_{k=1}^{m+1} |x|^{2k-2} v_k(x)$ for harmonic functions $v_1,\dots,v_m,v_{m+1}$, then we should have

$$\begin{aligned}\Delta u(x) &= \sum_{k=1}^{m} |x|^{2k-2} h_k(x) = \sum_{k=1}^{m+1} \Delta\left(|x|^{2k-2} v_k(x)\right) \\ &= \sum_{k=1}^{m} |x|^{2k-2}\left(4k(k-1+n/2)v_{k+1}(x) + 4k\sum_{j=1}^{n} x_j D_j v_{k+1}(x)\right),\end{aligned}$$

so that

$$h_k(x) = 4k(k-1+n/2)v_{k+1}(x) + 4k\sum_{j=1}^{n} x_j D_j v_{k+1}(x). \tag{4.1}$$

In fact, this is true for

$$v_{k+1}(x) = \frac{1}{4k}\int_0^1 r^{k-2+n/2} h_k(rx)\,dr.$$

Here note that v_{k+1} is harmonic in G and satisfies (4.1). Finally we have only to consider $v_1(x) = u(x) - \sum_{k=2}^{m+1} |x|^{2k-2} v_k(x)$, which is harmonic in G.

By an application of Almansi's theorem, we can show the following representation for polyharmonic functions on the balls; in the harmonic case, this is nothing but Poisson integral formula.

THEOREM 4.1. *Let u be polyharmonic of order m in $B(0,R)$. If $0<|x|<r<R$, then*

$$u(x) = \frac{(-1)^{m-1}}{\omega_n (m-1)!}(r^2-|x|^2)^m \int_{\mathbf{S}} \left(\frac{\partial}{\partial r^2}\right)^{m-1}\left(u(r\Theta)\frac{r^{n-2}}{|x-r\Theta|^n}\right) d\Theta.$$

PROOF. We show only the case $m=2$. In this case, we can write $u(x) = h_1(x) + |x|^2 h_2(x)$ for harmonic functions h_1 and h_2 on $B(0,R)$. If $|x|<r_1<r<r_2$, then

$$-\frac{{r_2}^2-|x|^2}{{r_1}^2-{r_2}^2}\frac{{r_1}^2-|x|^2}{\omega_n}\int_{\mathbf{S}}\frac{r_1^{n-2}u(r_1\Theta)}{|x-r_1\Theta|^n}d\Theta$$

$$
\begin{aligned}
&+\frac{{r_1}^2-|x|^2}{{r_1}^2-{r_2}^2}\frac{{r_2}^2-|x|^2}{\omega_n}\int_{\mathbf{S}}\frac{r_2^{n-2}u(r_2\Theta)}{|x-r_2\Theta|^n}d\Theta \\
= \quad &-\frac{{r_2}^2-|x|^2}{{r_1}^2-{r_2}^2}\frac{{r_1}^2-|x|^2}{\omega_n}\int_{\mathbf{S}}\frac{r_1^{n-2}}{|x-r_1\Theta|^n}[h_1(r_1\Theta)+{r_1}^2h_2(r_1\Theta)]d\Theta \\
&+\frac{{r_1}^2-|x|^2}{{r_1}^2-{r_2}^2}\frac{{r_2}^2-|x|^2}{\omega_n}\int_{\mathbf{S}}\frac{r_2^{n-2}}{|x-r_2\Theta|^n}[h_1(r_2\Theta)+{r_2}^2h_2(r_2\Theta)]d\Theta \\
= \quad &-\frac{{r_2}^2-|x|^2}{{r_1}^2-{r_2}^2}[h_1(x)+{r_1}^2h_2(x)]+\frac{{r_1}^2-|x|^2}{{r_1}^2-{r_2}^2}[h_1(x)+{r_2}^2h_2(x)]=u(x),
\end{aligned}
$$

from which the required formula follows.

LEMMA 4.3. *If v is harmonic in $B(0,r)$ and $a>-n$, then*

$$\frac{1}{|B(0,r)|}\int_{B(0,r)}|y|^a v(y)dy=\frac{n}{n+a}v(0)r^a.$$

In fact, applying polar coordinates, we have

$$
\begin{aligned}
\frac{1}{|B(0,r)|}\int_{B(0,r)}|y|^a v(y)dy &= \frac{1}{\sigma_n r^n}\int_0^r r^a\left(\int_{\mathbf{S}}v(r\Theta)d\Theta\right)r^{n-1}dr \\
&= \frac{1}{\sigma_n r^n}[\omega_n v(0)]\int_0^r r^a r^{n-1}dr \\
&= nv(0)(a+n)^{-1}r^a.
\end{aligned}
$$

For a function u and $r>0$, set

$$A_1(u,r)=\frac{1}{|B(0,r)|}\int_{B(0,r)}u(y)dy$$

and, inductively,

$$A_{j+1}(u,r)=\frac{1}{r^{n+2j}}\int_0^r A_j(u,t)t^{n+2j-1}dt.$$

LEMMA 4.4. *If u is polyharmonic of order m in $B(0,r_0)$, then there exist constants c_j such that*

$$u(0)=\sum_{j=1}^m c_jA_j(u,r) \qquad \textit{whenever } 0<r<r_0.$$

PROOF. Suppose u is polyharmonic of order m in $B(0,r)$, and write by Lemma 4.2

$$u(x)=\sum_{k=1}^m |x|^{2k-2}h_k(x)$$

for harmonic functions $h_1,\ldots,h_m$ on $B(0,r_0)$. Lemma 4.3 gives

$$A_1(u,r) = n\sum_{k=1}^{m} a_k r^{2k-2} h_k(0)$$

with $a_k = 1/(n+2k-2)$. Hence we find

$$A_j(u,r) = n\sum_{k=1}^{m} b_{k,j} r^{2k-2} h_k(0)$$

with $b_{k,j} = a_k \cdot a_{k+1} \cdots a_{k+j-1}$. Since $u(0) = h_0(0)$, we now have

$$u(0) = \frac{1}{nA}\begin{vmatrix} A_1(u,r) & b_{2,1} & \cdots & b_{m,1} \\ A_2(u,r) & b_{2,2} & \cdots & b_{m,2} \\ \cdots & \cdots & \cdots & \cdots \\ A_m(u,r) & b_{2,m} & \cdots & b_{m,m} \end{vmatrix}$$

with

$$A = \begin{vmatrix} b_{1,1} & b_{2,1} & \cdots & b_{m,1} \\ b_{1,2} & b_{2,2} & \cdots & b_{m,2} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1,m} & b_{2,m} & \cdots & b_{m,m} \end{vmatrix}.$$

For this purpose, we should check $A \neq 0$; in fact, note that

$$\begin{aligned} A &= a_1 a_2 \cdots a_m \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_2 & a_3 & \cdots & a_{m+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_2 \cdots a_m & a_3 \cdots a_{m+1} & \cdots & a_{m+1} \cdots a_{m+m-1} \end{vmatrix} \\ &= a_1 a_2 \cdots a_m \\ &\quad \times \begin{vmatrix} 1 & 0 & \cdots & 0 \\ a_2 & (-2)a_2 a_3 & \cdots & (-2)a_m a_{m+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_2 \cdots a_m & (-2(m-1))a_2 \cdots a_{m+1} & \cdots & (-2(m-1))a_m \cdots a_{m+m-1} \end{vmatrix} \\ &= (a_1 a_2 \cdots a_m)(-2)(-4)\cdots(-2(m-1))(a_2 {a_3}^2 \cdots {a_m}^2 a_{m+1}) \\ &\quad \times \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_4 & a_5 & \cdots & a_{m+2} \\ \cdots & \cdots & \cdots & \cdots \\ a_4 \cdots a_{m+1} & a_5 \cdots a_{m+2} & \cdots & a_{m+2} \cdots a_{m+m-1} \end{vmatrix} \\ &= (a_1 a_2 \cdots a_m)(-2)(-4)\cdots(-2(m-1))(a_2 {a_3}^2 \cdots {a_m}^2 a_{m+1}) \\ &\quad \times (-2)^{m-2}(-4)^{m-3}\cdots(-2(m-1)) a_4 \cdots {a_{m+1}}^{m-2} {a_{m+2}}^{m-2} \cdots a_{m+m-1} \\ &= (-2)^{m-1}(-4)^{m-2}\cdots(-2(m-1)) a_1 {a_2}^2 \cdots {a_m}^m {a_{m+1}}^{m-1} \cdots a_{m+m-1} \neq 0. \end{aligned}$$

COROLLARY 4.1. *If u is polyharmonic of order m in $B(x, r_0)$, then*

$$|u(x)| \leqq MA_1(|u|, r) \qquad \text{whenever } 0 < r < r_0.$$

LEMMA 4.5. *If u is polyharmonic of order m in $B(x, r_0)$, then*

$$(4.2) \qquad |\nabla_k u(x)| \leqq Mr^{-n-k} \int_{B(x,r)} |u(y)| dy \qquad \text{whenever } 0 < r < r_0,$$

where M is a positive constant independent of x and r.

PROOF. Suppose u is polyharmonic of order m in $B(0, r_0)$. Letting $D_i = \partial/\partial x_i$, we see from lemma 4.4 that

$$D_i u(0) = \sum_{j=1}^{m} c_j A_j(D_i u, r)$$

for $0 < r < r_0$. Note that

$$A_1(D_i u, r) = \frac{1}{|B(0, r)|} \int_{S(0,r)} u(y) \frac{x_i - y_i}{|x - y|} dS(y),$$

so that

$$|D_i u(0)| \leqq M \left(r^{-n} \int_{S(0,r)} |u(y)| dS(y) + r^{-n-1} \int_{B(0,r)} |u(y)| dy \right).$$

Multiplying both sides by r^{n+1} and integrating them with respect to r, we obtain

$$|D_i u(0)| \leqq Mr^{-n-1} \int_{B(0,r)} |u(y)| dy.$$

By considering translation, we see that

$$|D_i u(x)| \leqq Mr^{-n-1} \int_{B(x,r)} |u(y)| dy$$

whenever u is polyharmonic of order m in $B(x, r)$, where M is independent of x and r; in fact, M depends only on m. Thus it follows that

$$\int_{B(x,r)} |D_i u(y)| dy \leqq Mr^{-1} \int_{B(x,2r)} |u(y)| dy$$

whenever u is polyharmonic of order m in $B(x, 2r)$. Now, repeating this, we finally establish

$$\int_{B(x,r)} |\nabla_m u(y)| dy \leqq Mr^{-m} \int_{B(x,2^m r)} |u(y)| dy$$

whenever u is polyharmonic of order m in $B(x, 2^m r)$. This together with Corollary 4.1 gives (4.2).

If u is polyharmonic of order m in $B(x,r) \subseteqq \mathbf{H}$, then

$$\begin{aligned} |u(x)-u(z)| &\leqq |x-z| \sup_{y \in B(x,|x-z|)} |\nabla u(y)| \\ &\leqq M|x-z|^{1-n} \int_{B(x,2|x-z|)} |\nabla u(y)| dy \end{aligned}$$

for $z \in B(x,r/2)$. Hence, if in addition $B(x,r) \subseteqq \Gamma(\xi,a)$ for some $\xi \in \partial\mathbf{H}$, then

$$|u(x)-u(z)| \leqq M \int_{B(x,r)} |\nabla u(y)| y_1^{1-n} dy$$

for $z \in B(x,r/2)$. This can be generalized as follows :

LEMMA 4.6. *For $x \in \mathbf{H}$, $\xi \in \partial\mathbf{H}$ and $e = (1,0,...,0)$, let $x^* = \xi + |x-\xi|e$, $X(t) = x + t(x^*-x)$ and set*

$$E(x,x^*) = \bigcup_{0<t<1} B(X(t), X_1(t)/2).$$

If u is polyharmonic of order m in $\mathbf{H}$, then

$$|u(x)-u(x^*)| \leqq M \int_{E(x,x^*)} |\nabla u(y)| y_1^{1-n} dy \tag{4.3}$$

PROOF. By mean value theorem we have

$$\begin{aligned} |u(x)-u(x^*)| &\leqq |x-x^*| \int_0^1 |\nabla u(X(t))| dt \\ &\leqq M|x-x^*| \int_0^1 \left(\frac{1}{|B_t|} \int_{B_t} |\nabla u(y)| dy \right) dt, \end{aligned}$$

where $B_t = B(X(t), X_1(t)/2)$. Then note that $x_1 < X_1(t) < |x-\xi|$ and

$$|y-x| < t|x-x^*| + X_1(t)/2 < 2t|x-\xi| + X_1(t)/2 < (5/2)X_1(t) < 5y_1$$

for $y \in B_t$. Moreover, $(2/3)y_1 < X_1(t) = x_1 + t(x_1^* - x_1) < 2y_1$, so that the length of such t is at most $\max\{1, (4/3)y_1/(x_1^*-x_1)\}$. Then

$$|x-x^*| \max\{1, (4/3)y_1/(x_1^*-x_1)\} \leqq My_1,$$

so that Fubini's theorem gives

$$|u(x)-u(x^*)| \leqq M \int_{E(x,x^*)} |\nabla u(y)| y_1^{1-n} dy,$$

as required.

We say that a function u is polyharmonic in $\mathbf{H}$ if it is polyharmonic of order m in $\mathbf{H}$ for some positive integer m.

THEOREM 4.2. *Let $-1 < \beta < p-1$, and u be a polyharmonic function on $\mathbf{H}$ which satisfies*

$$\int_G \Phi_p(|\nabla u(x)|)x_1^\beta dx < \infty \qquad \text{for any bounded open set } G \subseteqq \mathbf{H}. \tag{4.4}$$

Define

$$\kappa(r) = \left(\int_r^1 [t^{n-p+\beta}\varphi(t^{-1})]^{-1/(p-1)}t^{-1}dt\right)^{1-p}$$

for $0 < r \leqq 1/2$; set $\kappa(r) = \kappa(1/2)$ for $r > 1/2$. For $\gamma \geqq 1$, define $h_\gamma(r) = \kappa(r^\gamma)$.

(i) *If $n-p+\beta > 0$ and $\gamma > 1$, then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $H_{h_\gamma}(E) = 0$ and u has a T_γ-limit at any $\xi \in \partial\mathbf{H} - E$.*

(ii) *If $n-p+\beta > 0$, then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $C_{1-\beta/p,\Phi_p}(E) = 0$ and u has a nontangential limit at any $\xi \in \partial\mathbf{H} - E$.*

(iii) *If $n-p+\beta = 0$, then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $C_{1-\beta/p,\Phi_p}(E) = 0$ and u has a T_∞-limit at any $\xi \in \partial\mathbf{H} - E$.*

(iv) *If $n-p+\beta < 0$, then u has a limit at any $\xi \in \partial\mathbf{H}$.*

PROOF. Consider the sets

$$E_1 = \left\{x \in \partial\mathbf{H} : \int_{B_+(x,1)} |x-y|^{1-n}f(y)dy = \infty\right\}$$

and

$$E_2 = \left\{x \in \partial\mathbf{H} : \limsup_{r\to 0}\ h_\gamma(r)^{-1}\int_{B_+(x,r)} \Phi_p(f(y))y_1^\beta dy > 0\right\},$$

where $f(y) = |\nabla u(y)|$. Here note that $C_{1-\beta/p,\Phi_p}(E_1) = 0$ by Lemma 3.3 and $H_{h_\gamma}(E_2) = 0$. With the aid of Lemmas 3.1 and 3.2, it suffices to show that u has a finite T_γ-limit at every $\xi \in \partial\mathbf{H} - (E_1 \cup E_2)$.

Let $x \in T_\gamma(\xi, a)$ and $x_0 = \xi + r_0 e$. Then we have by (4.3)

$$\begin{aligned} |u(x) - u(x_0)| &\leqq M\int_{E(x,x_0)} f(y)y_1^{1-n}dy \\ &= M\int_{E(x,x_0)-B(\xi,2|x-\xi|)} f(y)y_1^{1-n}dy \\ &\quad + M\int_{E(x,x_0)\cap B(\xi,2|x-\xi|)} f(y)y_1^{1-n}dy = u_1(x) + u_2(x). \end{aligned}$$

Since $y_1 > M(x_1 + |x-y|)$ for $y \in E(x, x_0)$, we find

$$u_1(x) \leqq M\int_{B_+(\xi,2r_0)} f(y)|\xi - y|^{1-n}dy.$$

Moreover, by Hölder's inequality we have

$$\begin{aligned}
u_2(x) &\leqq M \int_{\{y \in E(x,x_0) \cap B(\xi, 2|x-\xi|) : f(y) > y_1^{-\varepsilon}\}} f(y) y_1^{1-n} dy \\
&\quad + M \int_{\{y \in E(x,x_0) \cap B(\xi, 2|x-\xi|) : f(y) < y_1^{-\varepsilon}\}} f(y) y_1^{1-n} dy \\
&\leqq M \left(\int_{B(x,3|x-\xi|)} [(x_1 + |x-y|)^{1-n-\beta/p} \varphi((x_1+|x-y|)^{-\varepsilon})^{-1/p}]^{p'} dy \right)^{1/p'} \\
&\quad \times \left(\int_{B(\xi,2|x-\xi|)} \Phi_p(f(y)) y_1^{\beta} dy \right)^{1/p} + M \int_{B(\xi,2|x-\xi|)} |x-y|^{1-n-\varepsilon} dy \\
&\leqq M\kappa(x_1) \left(\int_{B(\xi,2|x-\xi|)} \Phi_p(f(y)) y_1^{\beta} dy \right)^{1/p} + M|x-\xi|^{1-\varepsilon}
\end{aligned}$$

for $0 < \varepsilon < 1$. If $\xi \in \partial\mathbf{H} - (E_1 \cup E_2)$, then we see that u is bounded on $T_\gamma(\xi, a) \cap B(\xi, r_0)$, so that we can find $X_j = \xi + r_j e$ tending to ξ for which $u(X_j)$ has a finite limit ℓ as $j \to \infty$. We replace x_0 by X_j and infer that

$$\lim_{j\to\infty} \left(\sup_{x \in T_\gamma(\xi,a) \cap B(\xi, r_j)} |u(x) - u(X_j)| \right) = 0.$$

Thus it follows that $u(x)$ has a finite limit ℓ as $x \to \xi$ along the sets $T_\gamma(\xi, a)$, and hence the proof is completed.

LEMMA 4.7. *Let $\xi \in \partial\mathbf{H}$ and u be polyharmonic in $\mathbf{H}$. If*

$$\int_{\Gamma(\xi,a)\cap B(\xi,1)} |\xi - y|^{k-n} |\nabla^k u(y)| dy < \infty$$

for all $a > 0$, then

$$\lim_{x \to \xi, x \in \Gamma(\xi,a)} |x-\xi|^k |\nabla^k u(x)| = 0$$

for all $a > 0$.

In fact, by Lemma 4.5, we have only to find that

$$\begin{aligned}
|x-\xi|^k |\nabla^k u(x)| &\leqq M|x-\xi|^{k-n} \int_{B(x,|x-\xi|/2)} |\nabla^k u(y)| dy \\
&\leqq M \int_{B(x,|x-\xi|/2)} |\xi - y|^{k-n} |\nabla^k u(y)| dy
\end{aligned}$$

whenever $x \in \Gamma(\xi, a)$.

Lemma 4.6 can be generalized as follows.

Lemma 4.8. *For $x \in \mathbf{H}$, let $x^* = \xi + |x-\xi|e$ and $X(t) = x^* + t(x - x^*)$. If u is polyharmonic in $\mathbf{H}$ and $U(t) = u(X(t))$, then*

$$\left| U(1) - U(0) - \frac{U'(0)}{1!} - \cdots - \frac{U^{(k-1)}(0)}{(k-1)!} \right| \leqq M \int_{E(x,x^*)} |\nabla^k u(y)| y_1^{k-n} dy.$$

Theorem 4.2′. *Let $-1 < \beta < p-1$, and u be a polyharmonic function on $\mathbf{H}$ which satisfies*

$$\int_G \Phi_p(|\nabla^k u(x)|) x_1^\beta dx < \infty \qquad \text{for any bounded open set } G \subseteqq \mathbf{H}. \tag{4.5}$$

(i) *If $n-kp+\beta > 0$ and $\gamma > 1$, then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $H_{h_\gamma}(E) = 0$ and u has a T_γ-limit at any $\xi \in \partial\mathbf{H} - E$, where $h_\gamma(r) \sim r^{\gamma(n-kp+\beta)}\varphi(t^{-1})$ as $r \to 0$.*

(ii) *If $n-kp+\beta > 0$, then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $C_{k-\beta/p,\Phi_p}(E) = 0$ and u has a nontangential limit at any $\xi \in \partial\mathbf{H} - E$.*

(iii) *If $n-kp+\beta = 0$, then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $C_{k-\beta/p,\Phi_p}(E) = 0$ and u has a T_∞-limit at any $\xi \in \partial\mathbf{H} - E$.*

(iv) *If $n-kp+\beta < 0$, then u has a limit at any $\xi \in \partial\mathbf{H}$.*

Proof. As in the proof of Theorem 4.2, consider the sets

$$E_1 = \left\{ x \in \partial\mathbf{H} : \int_{B_+(x,1)} |x-y|^{k-n} f(y) dy = \infty \right\}$$

and

$$E_2 = \left\{ x \in \partial\mathbf{H} : \limsup_{r \to 0} \, [h(r)]^{-1} \int_{B_+(x,r)} \Phi_p(f(y)) y_1^\beta dy > 0 \right\},$$

where $f(y) = |\nabla^k u(y)|$ and $h(r) = \kappa(r^\gamma)$ with

$$\kappa(r) = \left(\int_r^1 t^{-(n-kp+\beta)/(p-1)} dt/t \right)^{1-p}$$

for $0 < r \leqq 1/2$; set $\kappa(r) = \kappa(1/2)$ for $r > 1/2$. Note that $C_{k-\beta/p,\Phi_p}(E_1) = 0$ by Lemma 3.3 and $H_h(E_2) = 0$. With the aid of Lemmas 3.1 and 3.2, it suffices to show that u has a finite T_γ-limit at every $\xi \in \partial\mathbf{H} - (E_1 \cup E_2)$.

Let $x \in T_\gamma(\xi, a)$ and $x_0 = \xi + r_0 e$. Then we have by Lemma 4.8

$$\begin{aligned} \left| u(x) - u(x_0) - \cdots - \sum_{|\lambda|=k-1} \frac{(x-x_0)^\lambda}{\lambda!} D^\lambda u(x_0) \right| &\leqq M \int_{E(x,x_0)} f(y) y_1^{k-n} dy \\ &= M \int_{E(x,x_0) - B(\xi, 2|x-\xi|)} f(y) y_1^{k-n} dy \\ &\quad + M \int_{E(x,x_0) \cap B(\xi, 2|x-\xi|)} f(y) y_1^{k-n} dy \\ &= u_1(x) + u_2(x). \end{aligned}$$

Since $y_1 > M(x_1 + |x - y|)$ for $y \in E(x, x_0)$, we find

$$u_1(x) \leqq M \int_{B(\xi,2r_0)} f(y)|\xi - y|^{k-n} dy.$$

Moreover, by using Hölder's inequality, as in the proof of Theorem 4.2, we have

$$u_2(x) \leqq M\kappa(x_1) \left(\int_{B(\xi,2|x-\xi|)} \Phi_p(f(y)) y_1^\beta dy \right)^{1/p} + M|x - \xi|^{k-\varepsilon}$$

for $0 < \varepsilon < k$. If $\xi \in \partial\mathbf{H} - (E_1 \cup E_2)$, then we see that u is bounded on $T_\gamma(\xi, a) \cap B(\xi, r_0)$, so that we can find $X_j = \xi + r_j e$ tending to ξ for which $u(X_j)$ has a finite limit ℓ as $j \to \infty$. Since $\xi \in \mathbf{H} - E_1$, Lemma 2.2 implies that

$$\int_{\Gamma(\xi,a)\cap B(\xi,1)} |\xi - y|^{k-n} |\nabla^k u(y)| dy < \infty$$

for $k = 1, ..., m - 1$. Hence Lemma 4.7 gives

$$\lim_{j\to\infty} |X_j - \xi|^k |\nabla^k u(X_j)| = 0.$$

We replace x_0 by X_j and infer that

$$\lim_{j\to\infty} \left(\sup_{x \in T_\gamma(\xi,a)\cap B(\xi,r_j)} |u(x) - u(X_j)| \right) = 0.$$

Thus it follows that $u(x)$ has a finite limit ℓ as $x \to \xi$ along the sets $T_\gamma(\xi, a)$.

If $u(x) = P_{x_1} * f(x')$ with $f \in \Lambda_\alpha^{p,p}(\mathbf{R}^{n-1})$, then

$$\int_{\mathbf{R}_+^{n+1}} |\nabla^\ell u(x)|^p x_1^\beta dx < \infty,$$

where $\beta = (\ell - \alpha)p - 1$ for the smallest integer ℓ greater than α (see Chapter 7.5).

Corollary 4.1. *Let $u(x) = P_{x_1} * f(x')$ with $f \in \Lambda_\alpha^{p,p}(\mathbf{R}^{n-1})$. Suppose α is not an integer. Then :*

(i) *If $n - \alpha p - 1 > 0$ and $\gamma > 1$, then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $H_{\gamma(n-\alpha p-1)}(E) = 0$ and u has a T_γ-limit at any $\xi \in \partial\mathbf{H} - E$.*

(ii) *If $n - \alpha p - 1 > 0$, then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $C_{\alpha+1/p,p}(E) = 0$ and u has a nontangential limit at any $\xi \in \partial\mathbf{H} - E$.*

(iii) *If $n - \alpha p - 1 = 0$, then there exists a set $E \subseteqq \partial\mathbf{H}$ such that $C_{\alpha+1/p,p}(E) = 0$ and u has a T_∞-limit at any $\xi \in \partial\mathbf{H} - E$.*

(iv) *If $n - \alpha p - 1 < 0$, then u has a limit at any $\xi \in \partial\mathbf{H}$.*

In case α is a positive integer $\ell - 1$, then

$$\int_{\mathbf{R}_+^{n+1}} |\nabla^\ell u(x)|^p x_1^{\beta'} dx < \infty$$

for every β' such that $-1 < \beta' < \beta = p - 1$, so that the exceptional set in Corollary 4.1 (i) - (iii) should have Hausdorff dimension at most $n - \alpha p - 1$.

Proposition 4.1. *Let $-1 < \beta < p - 1$ and K be a compact subset of $\partial\mathbf{H}$ such that $C_{1-\beta/p,\Phi_p}(K) = 0$. Then there exists a harmonic function u on $\mathbf{H}$ which satisfies*

$$\int_{\mathbf{H}} \Phi_p(|\nabla u(x)| x_1^{\beta/p}) dx < \infty \tag{4.6}$$

and

$$\lim_{x \to \xi, x \in \mathbf{H}} u(x) = \infty \qquad \text{for any } \xi \in K.$$

Proof. Since $C_{1-\beta/p,\Phi_p}(K) = 0$, we can find a nonnegative function f with compact support which satisfies

$$\int_{\mathbf{R}^n} \Phi_p(f(y)) dy < \infty$$

and

$$g_{1-\beta/p} * f(\xi) = \infty \qquad \text{for every } \xi \in K.$$

In view of Theorems 5.1 and 7.1 in Chapter 7, we see that $u(x) = P_{x_1} * [g_{1-\beta/p} * f(0, x')]$ satisfies

$$\int_{\mathbf{H}} |\nabla u(x)|^q x_1^{(1-\alpha)q} dx \leqq M \int |f(y)|^q dy$$

with $\alpha = 1 - \beta/p$, when $1 < q < \infty$ and $-1 < q(1 - \alpha) < q - 1$. Hence, applying the usual interpolation techniques (cf. the proof of Lemma 3.3), we have

$$\int_{\mathbf{H}} \Phi_p(|\nabla u(x)| x_1^{\beta/p}) dx \leqq M \int \Phi_p(f(y)) dy,$$

which proves (4.6). Clearly, u has an infinite limit at every $\xi \in K$.

Remark 4.1. One sees easily that (4.6) implies (4.4).

8.5 Monotone functions

Let G be an open set in $\mathbf{R}^n$. A continuous function u on G is said to be monotone (in the sense of Lebesgue) if

$$\max_{\overline{D}} u = \max_{\partial D} u \qquad \text{and} \qquad \min_{\overline{D}} u = \min_{\partial D} u$$

hold for any relatively compact open subset D of G. Clearly harmonic functions on G are monotone. Further, if f is continuous and monotone on the interval $[0, \infty)$, then

$$u(x) = f(|x - \xi|)$$

is monotone on $\mathbf{H}$ for any $\xi \in \partial \mathbf{H}$.

THEOREM 5.1. *If $u \in C(G) \cap BL_1(L^p_{loc}(G))$ is a weak solution of the nonlinear Laplace operator Δ_p, then it is monotone in G.*

PROOF. Let D be a domain with compact closure in G, and $\max_{\partial D} u < M$. Then $u - u \wedge M$ vanishes near the boundary ∂D, so that

$$\int_D |\nabla u|^{p-2} \nabla u \cdot (\nabla u - \nabla(u \wedge M)) dx = 0.$$

Here note that $0 \leqq \nabla u \cdot (\nabla u - \nabla(u \wedge M)) \leqq |\nabla u|^2$, so that

$$\nabla u = \nabla(u \wedge M) \qquad \text{a.e. on } D.$$

Hence it follows that $u - u \wedge M$ is constant on D and, moreover, the constant must be zero. Thus $u = u \wedge M \leqq M$ on D, and the maximum property is satisfied.

The minimum principle is proved similarly.

THEOREM 5.2. *If u is monotone in $B(x, 2r)$, then*

$$|u(x) - u(y)|^p \leqq M r^{p-n} \int_{B(x,2r)} |\nabla u(z)|^p dz$$

whenever $p > n - 1$ and $y \in B(x, r)$.

PROOF. Let $y \in B(x, r)$, and we may assume that $u(y) > u(x)$. If $|x - y| < t < 2r$, then monotonicity implies the existence of x_t and $y_t \in S(x, t)$ for which $u(x_t) \leqq u(x) < u(y) \leqq u(y_t)$. Moreover, we see from Theorem 7.5 in Chapter 6 that

$$|u(x_t) - u(y_t)|^p \leqq M t^{p-(n-1)} \int_{S(x,t)} |\nabla u|^p dS.$$

Hence

$$|u(x) - u(y)|^p \leqq M t^{p-(n-1)} \int_{S(x,t)} |\nabla u|^p dS$$

for $|x - y| < t < 2r$. Integrating both sides with respect to $t \in (r, 2r)$, we obtain the required inequality.

THEOREM 5.3. *Let $-1 < \beta < p - 1$, $p > n - 1$ and u be a monotone function on $\mathbf{H}$ which satisfies*

$$\int_G |\nabla u(x)|^p x_1^\beta dx < \infty \qquad \text{for any bounded open set } G \subseteqq \mathbf{H}.$$

Then :

(i) *If* $n-p+\beta > 0$ *and* $\gamma > 1$, *then there exists a set* $E \subseteqq \partial \mathbf{H}$ *such that* $H_{h_\gamma}(E) = 0$ *and* u *has a* T_γ*-limit at any* $\xi \in \partial \mathbf{H} - E$, *where* $h_\gamma(r) \sim r^{\gamma(n-p+\beta)}\varphi(r^{-1})$ *as* $r \to 0$.

(ii) *If* $n-p+\beta > 0$, *then there exists a set* $E \subseteqq \partial \mathbf{H}$ *such that* $C_{1-\beta/p,p}(E) = 0$ *and* u *has a nontangential limit at any* $\xi \in \partial \mathbf{H} - E$.

(iii) *If* $n-p+\beta = 0$, *then there exists a set* $E \subseteqq \partial \mathbf{H}$ *such that* $C_{1-\beta/p,p}(E) = 0$ *and* u *has a* T_∞*-limit at any* $\xi \in \partial \mathbf{H} - E$.

(iv) *If* $n-p+\beta < 0$, *then* u *has a limit at any* $\xi \in \partial \mathbf{H}$.

Proof. As in the proof of Theorem 4.2, consider the sets

$$E_1 = \left\{ x \in \partial \mathbf{H} : \int_{B_+(x,1)} |x-y|^{1-n} f(y)dy = \infty \right\}$$

and

$$E_2 = \left\{ x \in \partial \mathbf{H} : \limsup_{r \to 0} \; h(r)^{-1} \int_{B_+(x,r)} \Phi_p(f(y)) y_1^\beta dy > 0 \right\},$$

where $f(y) = |\nabla u(y)|$ and $h(r) = \kappa(r^\gamma)$ with

$$\kappa(r) = \left(\int_r^1 [t^{n-p+\beta}\varphi(t^{-1})]^{-1/(p-1)} t^{-1} dt \right)^{1-p}$$

for $0 < r \leqq 1/2$; set $\kappa(r) = \kappa(1/2)$ for $r > 1/2$. As in the proof of Theorem 4.2, we have only to show that u has a finite T_γ-limit at every $\xi \in \partial \mathbf{H} - (E_1 \cup E_2)$.

For $\xi \in \partial \mathbf{H} - (E_1 \cup E_2)$, write

$$\begin{aligned} u(x) &= c \sum_{j=1}^n \int (x_j - y_j)|x-y|^{-n} D_j \overline{u}(y) dy + A \\ &= c \sum_{j=1}^n \int_{B(x,x_1/2)} (x_j - y_j)|x-y|^{-n} D_j \overline{u}(y) dy \\ &\quad + c \sum_{j=1}^n \int_{B(\xi,2|\xi-x|)-B(x,x_1/2)} (x_j - y_j)|x-y|^{-n} D_j \overline{u}(y) dy \\ &\quad + c \sum_{j=1}^n \int_{\mathbf{R}^n - B(\xi,2|\xi-x|)} (x_j - y_j)|x-y|^{-n} D_j \overline{u}(y) dy + A \\ &= u_1(x) + u_2(x) + u_3(x) + A, \end{aligned}$$

where $\overline{u}$ is the symmetric extension of u with respect to $\partial \mathbf{H}$. We see from Lebesgue's dominated convergence theorem that

$$\lim_{x \to \xi} u_3(x) = c \sum_{j=1}^n \int (\xi_j - y_j)|\xi - y|^{-n} D_j \overline{u}(y) dy.$$

In view of the proof of Theorem 4.2, we have

$$\begin{aligned} |u_2(x)| &\leqq |c| \int_{B(\xi,2|\xi-x|)-B(x,x_1/2)} |x-y|^{1-n}\overline{f}(y)dy \\ &\leqq M\kappa(x_1)\left(\int_{B(\xi,2|x-\xi|)} [\overline{f}(y)]^p y_1^\beta dy\right)^{1/p} + M|x-\xi|^{1-\varepsilon}, \end{aligned}$$

where $0 < \varepsilon < 1$ and $\overline{f}(y) = |\nabla \overline{u}(y)|$. Hence

$$\lim_{x\to\xi, x\in T_\gamma(\xi,a)} u_2(x) = 0.$$

Note that

$$|u_1(x)| \leqq M \int_{B(x,x_1/2)} |x-y|^{1-n}\overline{f}(y)dy.$$

As in the proof of Theorem 4.4 in Chapter 5, we can find a set $F \subseteqq \mathbf{H}$ for which

$$\lim_{x\to\xi, x\in \mathbf{H}-F} u_1(x) = 0$$

and

$$\lim_{j\to\infty} [h(2^{-j})]^{-1} C_{1,p,\beta}(F \cap B(\xi,2^{-j}); B(\xi,2^{-j+2})) = 0.$$

Let $A_j = B(\xi,2^{-j+1}) - B(\xi,2^{-j})$. If $x \in A_j \cap T_\gamma(\xi,a)$, then we see from Lemma 3.1 that

$$C_{1,p,\beta}(B(x,x_1/4); B(\xi,2^{-j+2})) \geqq Mh(2^{-j}).$$

Consequently, for $x \in A_j \cap T_\gamma(\xi,a)$, there exists $y \in B(x,x_1/4)$ which does not belong to F. On the other hand, Theorem 5.2 implies that

$$|u(x)-u(y)|^p \leqq M x_1^{p-n} \int_{B(x,x_1/2)} f(z)^p dz$$

which proves

$$\lim_{x\to\xi, x\in T_\gamma(\xi,a)} u_1(x) = 0.$$

Thus it follows that $u(x)$ has a finite limit as $x \to \xi$ along the sets $T_\gamma(\xi,a)$.

Index

General references

[1] Books on measure theory

1. H. Federer, Geometric measure theory, Springer-Verlag, 1969.
2. P. Halmos, Measure theory, Van Nostrand, New York, 1950.

[2] Books on the classical potential theory

1. M. Brelot, Lectures on potential theory, Tata Institute No. 19, 1967.
2. M. Brelot, Éléments de la théorie classique du potentiel, 4^e édition, Centre de Documentation Universitaire, Paris, 1969.
3. M. Brelot, On topologies and boundaries in potential theory, Lecture Notes in Math. 175, Springer-Verlag, 1971.
4. L. Carleson, Selected problems on exceptional sets, Van Nostrand, 1967.
5. J. L. Doob, Classical potential theory and its probabilistic counterpart, Springer, 1984.
6. W. K. Hayman and P. B. Kennedy, Subharmonic functions Vol. 1, Academic Press, 1976.
7. W. K. Hayman, Subharmonic functions Vol. 2, Academic Press, 1989.
8. L. L. Helms, Introduction to potential theory, Wiley-Interscience, 1969.
9. O. D. Kellog, Foundations of potential theory, Springer, 1967.
10. M. Kishi, Potential theory (In Japanese), Morikita, 1974.
11. N. S. Landkof, Foundations of modern potential theory, Springer, 1972.
12. M. Tsuji, Potential theory in modern function theory, Maruzen, 1959.
13. J. Wermer, Potential theory, Springer Lecture Notes 408, 1974.

[3] Books on L^p-potential theory

1. L. C. Evans and R. F. Gariepy, Measure theory and fine properties of functions, CRC Press, 1992.

2. V. G. Maz'ja, Sobolev spaces, springer-Verlag, 1985.

3. S. L. Sobolev, Applications of functional analysis in mathematical physics, Amer. Math. Soc., Transl., Math. Mono. 7 (1963).

4. E. M. Stein. Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, 1970.

5. W. P. Ziemer, Weakly differentiable functions, Springer-Verlag, 1989.

[4] Lecture Notes by M. Ohtsuka

1. M. Ohtsuka, Modern theory of Newtonian potentials, Lecture Notes at Hiroshima University, 1970-1971 (revised in 1977).

2. M. Ohtsuka, Harmonic and superharmonic functions, Lecture Notes at Hiroshima University, 1971-1972.

3. M. Ohtsuka, Extremal length and precise functions in 3-space, Lecture Notes at Hiroshima University, 1972.

Bibliography

[1] D. R. Adams, Lectures on L^p-potential theory, Department of Mathematics, University of Umeå, 1981.

[2] D. R. Adams and N. G. Meyers, Thinness and Wiener criteria for non-linear potentials, Indiana Univ. Math. J. 22 (1972), 169-197.

[3] D. R. Adams and N. G. Meyers, Bessel potentials. Inclusion relations among classes of exceptional sets, Indiana Univ. Math. J. 22 (1973), 873-905.

[4] D. R. Adams and J. C. Polking, The equivalence of two definition of capacity, Proc. Amer. Math. Soc. 37 (1973), 529-534.

[5] H. Aikawa, On the behavior at infinity of non-negative superharmonic functions in a half space, Hiroshima Math. J. 11 (1981), 425-441.

[6] H. Aikawa, On the minimal thinness in a Lipschitz domain, Analysis 5 (1985), 347-382.

[7] H. Aikawa, On the Martin boundary of Lipschitz strips, J. Math. Soc. Japan 38 (1986), 527-541.

[8] H. Aikawa, On subharmonic functions in strips, Ann. Acad. Sci. Fenn. Ser. A I Math. 1 (1987), 119-134.

[9] H. Aikawa, On weighted Beppo Levi functions - Integral representations and behavior at infinity, Analysis 9 (1989), 323-346.

[10] H. Aikawa, Comparison of L^p-capacity and Hausdorff measure, Complex Variables 15 (1990), 223-232.

[11] H. Aikawa, Harmonic functions having no tangential limits, Proc. Amer. Math. Soc. 108 (1990), 457-464.

[12] H. Aikawa, Quasiadditivity of Riesz capacity, Math. Scand. 69 (1991), 15-30.

[13] H. Aikawa, Harmonic functions and Green potentials having no tangential limits, J. London Math. Soc. (2) 43 (1991), 125-136.

[14] H. Aikawa, Thin sets at the boundary, Proc. London Math. Soc. (3), 65 (1992), 357-382.

[15] V. Anandam, On the integral representation of superharmonic functions in R^n, Acad. Roy. Belg. Bull. Cl. Sci. (5) 61 (1980), 307-318.

[16] N. Aronszajn and K. T. Smith, Theory of Bessel potentials, Part 1, Ann. Inst. Fourier 11 (1961), 385-475.

[17] N. Aronszajn, F. Mulla and P. Steptycki, On spaces of potentials connected with L^p classes, Ann. Inst. Fourier 13 (1963), 211-306.

[18] A. A. Bagarshakyan, On angular boundary values of some class of functions, Izv. Akad. Nauk Armajan SSR Ser. Mat. 9 (1974), 433-445.

[19] A. F. Beardon, Montel's theorem for subharmonic functions and solutions of partial differential equations, Proc. Cambridge Philos. Soc. 69 (1971), 123-150.

[20] Bui Huy Qui and Y. Mizuta, On the existence of harmonic functions in L^p, J. Math. Soc. Japan 33 (1981), 505-507.

[21] A. P. Calderón, Lebesgue spaces of differentiable functions and distributions, Proc. sympos. Pure Math. IV, Amer. Math. Soc., 1961, 33-49.

[22] L. Carleson, Selected problems on exceptional sets, Van Nostrand, Princeton, 1967.

[23] G. Choquet, Theory of capacities, Ann. Inst. Fourier 5 (1953), 131-295.

[24] R. R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math. 51 (1974), 241-250.

[25] A. B. Cruzeiro, Convergence au bord pour les fonctions harmoniques dans R^d de la classe de Sobolev W_1^d, C. R. Acad. Sci. Paris 294 (1982), 71-74.

[26] A. M. Davie and B. Øksendal, Analytic capacity and differentiability properties of finely harmonic functions, Acta Math. 149 (1982), 127-152.

[27] J. Deny, Les potentiels d'énergie finie, Acta Math. 82 (1950), 107-183.

[28] J. Deny and J. L. Lions, Les espaces du type de Beppo Levi, Ann. Inst. Fourier 5 (1955), 305-370.

[29] J. R. Diederich, Natural limits for harmonic and superharmonic functions, Trans. Amer. Math. Soc. 224 (1976), 381-397.

[30] J. Edenhofer, Integraldarstellung einer m-polyharmonischen Fuktion, deren Funktionswerte und erste $m-1$ Normalableitungen auf einer Hyperphäre gegeben sind, Math. Nachr. 68 (1975), 105-113.

[31] M. Éssen and H. J. Jackson, On the covering properties of certain exceptional sets in a half space, Hiroshima Math. J. 10 (1980), 233-262.

[32] C. Fefferman, Convergence on almost every line for functions with gradient in $L^p(R^n)$, Ann. Inst. Fourier 24 (1974), 159-164.

[33] B. Fuglede, Extremal length and functional completion, Acta Math. 98 (1957), 171-219.

[34] B. Fuglede, On generalized potentials of functions in Lebesgue classes, Math. Scand. 8 (1960), 287-304.

[35] B. Fuglede, Le théorème du minimax et la théorie fine du potentiel, Ann. Inst. Fourier 15 (1965), 65-88.

[36] B. Fuglede, Fonctions BLD et fonctions finement surharmoniques, Séminaire de Théorie du Potentiel, No. 6, Lecture Notes in Math. 906, Springer, 1982.

[37] V. I. Gavrilof, On theorems of Beurling, Carleson and Tsuji on exceptional sets, Math. USSR-Sb. 23 (1974), 1-12.

[38] R. Harvey and J. C. Polking, Removable singularities of solutions of linear partial differential equations, Acta Math. 125 (1970), 39-56.

[39] R. Harvey and J. C. Polking, A notion of capacity which characterized removable singularities, Trans. Amer. Math. Soc. 169 (1972), 183-195.

[40] K. Hatano, Existence and Hölder continuity of derivatives of single layer Φ-potentials, Hiroshima Math. J. 13 (1983), 543-582.

[41] K. Hatano, Bessel capacity of symmetric generalized Cantor sets, Hiroshima Math. J. 17 (1987), 149-156.

[42] K. Hatano, A note on (α, p)-thinness of symmetric generalized Cantor sets, Mem. Fac. Educ. Shimane Univ. (Nat. Sci.) 22 (1988), 1-4.

[43] L. I. Hedberg, Removable singularities and condenser capacities, Ark. Mat. 12 (1974), 181-201.

[44] H. L. Jackson, On the boundary behavior of BLD functions and some applications, Acad. Roy. Belg. Bull. Cl. Sci. (5) 66 (1980), 223-239.

[45] R. Kaufman, Hausdorff measure, BMO, and analytic functions, Pacific J. Math. 102 (1982), 369-371.

[46] R. Kaufman and J.-M. G. Wu, Removable singularities for analytic or harmonic functions, Ark. Mat. 18 (1980), 107-116.

[47] N. Kenmochi and Y. Mizuta, The gradient of a convex function on a regular functional space and its potential theoretic properties, Hiroshima Math. J. 4 (1974), 743-763.

[48] N. Kenmochi and Y. Mizuta, Potential theoretic properties of the gradient of a convex function on a functional space, Nagoya Math. J. 59 (1975), 199-215.

[49] T. Kurokawa, A weighted norm inequality for potentials of order (m, k), J. Math. Kyoto Univ. 26-2 (1986), 203-211.

[50] T. Kurokawa, On weighted norm inequalities for Riesz potentials, Japan J. Math. 14 (1988), 261-274.

[51] T. Kurokawa, Riesz potentials, higher Riesz transforms and Beppo Levi spaces, Hiroshima Math. J. 18 (1988), 541-597.

[52] T. Kurokawa, Integral representation of smooth functions in weight classes and its application, Nagoya Math. J. 14 (1988), 1-11.

[53] T. Kurokawa, On hypersingular integrals, Osaka J. Math. 27 (1990), 721-738.

[54] T. Kurokawa, On modified singular integrals, Hiroshima Math. J. 23 (1993), 537-550.

[55] T. Kurokawa and Y. Mizuta, On the existence of boundary values of p-precise functions, Hiroshima Math. J. 7 (1977), 153-160.

[56] T. Kurokawa and Y. Mizuta, On the order at infinity of Riesz potentials, Hiroshima Math. J. 9 (1979), 533-545.

[57] J. Lelong-Ferrand, Étude au voisinage de la fronti¸'ere des fonctions surharmoniques positive dans un demi-espace, Ann. Sci. École Norm. sup. 66 (1949), 125-159.

[58] V. G. Maz'ja, Sobolev spaces, Springer-Verlag, 1985.

[59] N. G. Meyers, A theory of capacities for potentials in Lebesgue classes, Math. Scand. 8 (1970), 255-292.

[60] N. G. Meyers, Taylor expansion of Bessel potentials, Indiana Univ. Math. J. 23 (1974), 1043-1049.

[61] N. G. Meyers, Continuity properties of potentials, Duke Math. J. 42 (1975), 157-166.

[62] Y. Mizuta, Integral representations of Beppo Levi functions of higher order, Hiroshima Math. J. 4 (1974), 375-396.

[63] Y. Mizuta, On the existence of boundary values of Beppo Levi functions defined in the upper half space of R^n, Hiroshima Math. J. 6 (1976), 61-72.

[64] Y. Mizuta, On the limits of p-precise functions along lines parallel to the coordinate axes of R^n, Hiroshima Math. J. 6 (1976), 353-357.

[65] Y. Mizuta, On the existence of non-tangential limits of harmonic functions, Hiroshima Math. J. 7 (1977), 161-164.

[66] Y. Mizuta On the radial limits of Riesz potentials at infinity, Hiroshima Math. J. 7 (1977), 165-175.

[67] Y. Mizuta, On removable singularities for polyharmonic distributions, Hiroshima Math. J. 7 (1977), 827-832.

[68] Y. Mizuta, On the radial limits of potentials and angular limits of harmonic functions, Hiroshima Math. J. 8 (1978), 415-437.

[69] Y. Mizuta, Fine differentiability of Riesz potentials, Hiroshima Math. J. 8 (1978), 505-514.

[70] Y. Mizuta, Existence of various boundary limits of Beppo Levi functions of higher order, Hiroshima Math. J. 9 (1979), 717-745.

[71] Y. Mizuta, Existence of non-tangential limits of solutions of non-linear Laplace equation, Hiroshima Math. J. 10 (1980), 365-368.

[72] Y. Mizuta, Continuity of contractions in a functional Banach space, Hiroshima Math. J. 10 (1980), 381-384.

[73] Y. Mizuta, On the behavior at infinity of Green potentials in a half space, Hiroshima Math. J. 10 (1980), 607-613.

[74] Y. Mizuta, Boundary limits of Green potentials of order α, Hiroshima Math. 11 (1981), 111-123.

[75] Y. Mizuta, Semi-fine limits and semi-fine differentiability of Riesz potentials of functions in L^p, Hiroshima Math. J. 11 (1981), 515-524.

[76] Y. Mizuta, On the boundary behavior of superharmonic functions in a half space, J. Math. Soc. Japan 34 (1982), 269-278.

[77] Y. Mizuta, Minimally semi-fine limits of Green potentials of general order, Hiroshima Math. J. 12 (1982), 505-511.

[78] Y. Mizuta, On semi-fine limits of potentials, Analysis 2 (1982), 115-139.

[79] Y. Mizuta, On the behavior of harmonic functions near a hyperplane, Analysis 2 (1982), 203-218.

[80] Y. Mizuta, On the behaviour at infinity of superharmonic functions, J. London Math. Soc. (2) 27 (1983), 97-105.

[81] Y. Mizuta, On the behavior of potentials near a hyperplane, Hiroshima Math. J. 13 (1983), 529-542.

[82] Y. Mizuta, On the boundary limits of polyharmonic functions in a half space, Kodai Math. J. 7 (1984), 192-202.

[83] Y. Mizuta, On the boundary limits of harmonic functions with gradient in L^p, Ann. Inst. Fourier Grenoble 34 (1984), 99-109.

[84] Y. Mizuta, Study of the behavior of logarithmic potentials by means of logarithmically thin sets, Hiroshima Math. J. 14 (1984), 227-246.

[85] Y. Mizuta, On the behavior at infinity of logarithmic potentials, J. Math. Soc. Japan 36 (1984), 475-481.

[86] Y. Mizuta, On the growth of α-potentials in R^n and thinness of sets, Hiroshima Math. J. 15 (1985), 585-599.

[87] Y. Mizuta, On removability of sets for holomorphic and harmonic functions. J. Math. Soc. Japan 38 (1986), 509-513.

[88] Y. Mizuta, On the existence of limits along lines of Beppo Levi functions, Hiroshima Math. J. 16 (l986), 387-404.

[89] Y. Mizuta, On the existence of polyharmonic functions in Lebesgue classes, Proc. Japan Acad. 62 (1986), 300-303.

[90] Y. Mizuta, Boundary limits of Green potentials of general order, Proc. Amer. Math. Soc. 101 (1987), 131-135.

[91] Y. Mizuta, On the differentiability of Riesz potentials of functions, Hiroshima Math. J. 17 (1987), 355-359.

[92] Y. Mizuta, Boundary behavior of p-precise functions on a half space of R^n, Hiroshima Math J. 18 (1988), 73-94.

[93] Y. Mizuta, On the boundary limits of harmonic functions, Hiroshima Math. J. 18 (1988), 207-217.

[94] Y. Mizuta, On the boundary limits of Green potentials of functions, J. Math. Soc. Japan 40 (1988), 583-594.

[95] Y. Mizuta, Continuity properties of Riesz potentials and boundary limits of Beppo Levi functions, Math. Scand. 63 (1988), 238-260.

[96] Y. Mizuta, Integral representations of Beppo Levi functions and the existence of limits at infinity, Hiroshima Math. J. 19 (1989), 259-279.

[97] Y. Mizuta, On the existence of spherical limits of Beppo Levi functions, J. London Math. Soc. (2) 41 (1990), 89-99.

[98] Y. Mizuta, Boundary limits of locally n-precise functions, Hiroshima Math. J. 20 (1990), 109-126.

[99] Y. Mizuta, On the existence of weighted boundary limits of harmonic functions, Ann. Inst. Fourier Grenoble 40, 4 (1990), 811-833.

[100] Y. Mizuta, Boundary limits of harmonic functions in Sobolev-Orlicz classes, The Proceedings of ICPT 90 in Nagoya, 1991, 235-249.

[101] Y. Mizuta, Fine limits of potentials, Analysis 11 (1991), 345-381.

[102] Y. Mizuta, Growth properties of p-th hyperplane means of Green potentials in a half space, J. Math. Soc. Japan 44 (1992), 1-12.

[103] Y. Mizuta, Spherical means of Beppo Levi functions, Math. Nachr. 158 (1992), 241-262.

[104] Y. Mizuta, Continuity properties of potentials and Beppo-Levi-Deny functions, Hiroshima Math. J. 23 (1993), 79-153.

[105] Y. Mizuta and Bui Huy Qui, On the existence of non-tangential limits of polyharmonic functions, Hiroshima Math. J. 8 (1978), 409-414.

[106] T. Murai, On the behavior of functions with finite weighted Dirichlet integral near the boundary, Nagoya Math. J. 53 (1974), 83-101.

[107] A. Nagel, W. Rudin and J. H. Shapiro, Tangential boundary behavior of functions in Dirichlet-type spaces, Ann. of Math. 116 (1982), 331-360.

[108] M. M. Nicolesco, Recherches sur les fonctions polyharmoniques, Ann. Sci. École Norm. Sup. 52 (1935), 183-220.

[109] Yu. G. Reshetnyak, The concept of capacity in the theory of functions with generalized derivatives, Siberian Math. J. 10 (1969), 818-842.

[110] Yu. G. Reshetnyak, On the boundary behavior of functions with generalized derivatives, Siberian Math. J. 13 (1972), 285-290.

[111] P. J. Rippon, Semi-fine limits of Green potentials, Bull. London Math. Soc. 11 (1979), 259-264.

[112] J. Serrin, Local behavior of solutions of quasi-linear equations, Acta Math. 111 (1964), 247-302.

[113] M. L. Silverstein and R. L. Wheeden, Superharmonic functions on Lipschitz domains, Studia Math. 39 (1871), 191-198.

[114] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, 1970.

[115] D. C. Ullrich, Removable sets for harmonic functions, Michigan Math. J. 38 (1991), 467-473.

[116] N. X. Uy, Removable sets of analytic functions satisfying a Lipschitz condition, Ark. Mat. 17 (1979), 19-27.

[117] N. X. Uy, A removable sets for Lipschitz harmonic functions, Michigan Math. J. 37 (1990), 45-51.

[118] H. Wallin, Continuous functions and potential theory, Ark. Mat. 5 (1963), 55-84.

[119] H. Wallin, On the existence of boundary values of a class of Beppo Levi functions, Trans. Amer. Math. Soc. 120 (1965), 510-525.

[120] H. Wallin, Riesz potentials, k, p-capacity and p-modules, Mich. Math. J. 18 (1971), 257-263.

[121] J.-M. G. Wu, L^p-densities and boundary behaviors of Green potentials, Indiana Univ. Math. J. 28 (1979), 895-911.

[122] W. P. Ziemer, Extremal length as a capacity, Michigan Math. J. 17 (1970), 117-128.

[123] L. Ziomek, On the boundary behavior in the metric L^p of superharmonic functions, Studia Math. 29 (1967), 97-105.

[124] A. Zygmund, On a theorem of Marcinkiewicz concerning interpolation of operators, J. Math. Pures Appl. 35 (1956), 223-248.